

On small energy stabilization in the NLKG with a trapping potential

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Abstract

We consider a nonlinear Klein Gordon equation (NLKG) with short range potential with eigenvalues and show that in the contest of complex valued solutions the small standing waves are attractors for small solutions of the NLKG. This extends the results already known for the nonlinear Schrödinger equation and for the nonlinear Dirac equation. In addition, this extends a result of Bambusi and Cuccagna (which in turn was an extension of a result by Soffer and Weinstein) which considered only real valued solutions of the NLKG.

1 Introduction

We consider the following nonlinear Klein Gordon equation initial value problem in $(t, x) \in \mathbb{R} \times \mathbb{R}^3$:

$$\begin{cases} \dot{v} + Hu + m^2u + |u|^2u = 0 \text{ and } \dot{u} = v, \\ u(0) = u_0 \in H^1(\mathbb{R}^3, \mathbb{C}) \text{ and } v(0) = v_0 \in L^2(\mathbb{R}^3, \mathbb{C}). \end{cases} \quad (1.1)$$

where $H = -\Delta + V$ with $V \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$, the space of real valued and Schwartz functions.

We recall that if H has eigenvalues, then the solutions of the linear version of (1.1), that is with $|u|^2u$ omitted, are formed by various uncoupled oscillators and by scattering continuous modes. In particular, each eigenvalue can be associated to an invariant linear space in $H^1 \times L^2$. Such invariant spaces exist near the origin also for the nonlinear problem (1.1) in the form of topological disks which are tangent to the linear spaces at the origin. In the case of the nonlinear Schrödinger equation, [9], and of the nonlinear Dirac equation, [11], it has been shown that the union of these disks is an attractor for all small energy solutions. We will prove the same result for (1.1). This, as in [9, 11], is non trivial because one might have expected (1.1) to have complicated quasi periodic solutions as for discrete equations, e.g. [20].

A partial version of the problem addressed here has been considered in [28, 2] (for a further addition to the result in [28] see also [1]). In particular, in [2] it is shown in quite some generality that when u_0 and v_0 are real valued then any small energy solution $(u(t), v(t))$ scatters to the origin, that is $\rho = 0$ in Theorem 1.5 below. Here we generalize this because we consider u_0 and v_0 complex valued. In this latter case $(u(t), v(t))$ can scatter to small standing waves, which are always not real valued and therefore could not be seen in [28, 2]. Since the attracting set is more complicated than just a single point, the result we obtain here is substantially different than [2], and the proof is more elaborate. Obviously, our paper is motivated by our interest on discrete and continuous modes interactions. Indeed, if H has no eigenvalues, then the scattering to 0 of all small energy solutions is a standard consequence of the results on wave operators in Yajima [34].

Before further comments we introduce some notation, the hypotheses and our main results.
For $g, h : \mathbb{R}^3 \rightarrow \mathbb{C}^i$ ($i = 1, 2$ with $g = (g_1, g_2)$ and $h = (h_1, h_2)$ if $i = 2$), we use the real inner product

$$\langle g|h \rangle = \begin{cases} \operatorname{Re} \int_{\mathbb{R}^3} g(x) \overline{h(x)} dx, & \text{for } i = 1, \\ \operatorname{Re} \int_{\mathbb{R}^3} (g_1(x) \overline{h_1(x)} + g_2(x) \overline{h_2(x)}) dx, & \text{for } i = 2. \end{cases} \quad (1.2)$$

We introduce the Japanese bracket $\langle x \rangle := \sqrt{1 + |x|^2}$ and the spaces defined by the following norms:

$$\begin{aligned} L^{p,s}(\mathbb{R}^3, X) &\text{ defined with } \|u\|_{L^{p,s}(\mathbb{R}^3, X)} := \|\langle x \rangle^s u\|_{L^p(\mathbb{R}^3, X)}; \\ H^k(\mathbb{R}^3, X) &\text{ defined with } \|u\|_{H^k(\mathbb{R}^3, X)} := \|\langle x \rangle^k \widehat{u}\|_{L^2(\mathbb{R}^3, X)}, \text{ where } \widehat{u} \text{ the Fourier transform of } u; \\ \Sigma^k(\mathbb{R}^3, X) &:= L^{2,k}(\mathbb{R}^3, X) \cap H^k(\mathbb{R}^3, X) \text{ with } \|u\|_{\Sigma^k}^2 = \|u\|_{H^k(\mathbb{R}^3, X)}^2 + \|u\|_{L^{2,k}(\mathbb{R}^3, X)}^2, \end{aligned} \quad (1.3)$$

where $X = \mathbb{C}, \mathbb{C}^2$ or \mathbb{R} . We assume the following.

- (H1) $V \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$, where $\mathcal{S}(\mathbb{R}^3, \mathbb{R}) = \cap_{m \geq 0} \Sigma^m(\mathbb{R}^3, \mathbb{R})$ is the space of Schwartz functions.
- (H2) $\sigma_p(H)$ is formed by points $-m^2 < e_1 < e_2 < e_3 \cdots < e_n < 0$, where $\sigma_p(H)$ is the set of point spectrum of H . We also assume that all the eigenvalues have multiplicity 1. Moreover, 0 is neither an eigenvalue nor a resonance (that is, if $Hu = 0$ with $u \in C^\infty$ and $|u(x)| \leq C|x|^{-1}$ for a fixed C , then $u = 0$).
- (H3) Set $\omega_j = \sqrt{m^2 + e_j}$ and consider the smallest $N \in \mathbb{N}$ s.t. $N \min\{\omega_j \pm \omega_l : j \geq l\} \geq 2m$. We assume that $|\sum_{j=1}^n m_n \omega_n| \neq m$ for all $|\mathbf{m}| \leq 4N + 6$.
- (H4) The Fermi Golden Rule (FGR) holds: the expression

$$\sum_{L \in \Lambda} L \langle G_L | \delta(-\sigma_3 B - L) \sigma_3 G_L \rangle,$$

which is defined in the course of the paper (for $\Lambda \subset \mathbb{R}_+$ see under (6.15), for G_L see (6.19)) and is shown in Sect. 6.2 to be non-negative and to equal the l.h.s. of (6.22), is assumed here to satisfy inequality (6.22).

We introduce constants

$$\begin{aligned} \tilde{\omega} &= (\tilde{\omega}_1, \dots, \tilde{\omega}_{2n}) \text{ with } \tilde{\omega}_J = (-1)^{\kappa_J} \omega_J, \text{ and} \\ \kappa_J &= \begin{cases} 0 & \text{for } J = 1, \dots, n, \\ 1 & \text{for } J = n+1, \dots, 2n, \end{cases} \text{ we set } \omega_J = \begin{cases} \omega_J & \text{for } J \leq n, \\ \omega_{J-n} & \text{for } J > n. \end{cases} \end{aligned} \quad (1.4)$$

To each e_j we associate an eigenfunction ϕ_j . We choose them s.t. $\langle \phi_j | \phi_k \rangle = \delta_{jk}$. Since we can, we also choose the ϕ_j to be all real valued. To each ϕ_j we associate nonlinear bound states. For a proof of the following standard result see Appendix A in [9]. Notice that the real analyticity with respect to z is also immediate from the fact that the nonlinearity is analytic, see also [20]. Moreover, for the spaces Σ^t and the notation $D_X(v, \delta)$, we refer to Sect. 2.1.

Proposition 1.1 (Bound states). *There exists $a_0 > 0$ such that for every $j \in \{1, \dots, n\}$, and $\forall z \in D_{\mathbb{C}}(0, a_0)$, there is a unique $Q_{jz} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}) := \cap_{t \geq 0} \Sigma^t(\mathbb{R}^3, \mathbb{C})$, s.t.*

$$HQ_{jz} + |Q_{jz}|^2 Q_{jz} = E_{jz} Q_{jz} \quad , \quad Q_{jz} = z \phi_j + q_{jz}, \quad \langle q_{jz} | \phi_j \rangle = 0, \quad (1.5)$$

and s.t. we have for any $r \in \mathbb{N}$:

- (1) $(q_{Jz}, E_{Jz}) \in C^\omega(D_{\mathbb{C}}(0, a_0), \Sigma^r(\mathbb{R}^3, \mathbb{C}) \times \mathbb{R})$; and $q_{Jz} = z\widehat{q}_j(|z|^2)$, with $\widehat{q}_j(t^2) = t^2\widetilde{q}_j(t^2)$, for $\widetilde{q}_j(t) \in C^\omega((-a_0^2, a_0^2), \Sigma^r(\mathbb{R}^3, \mathbb{R}))$, and $E_{Jz} = E_j(|z|^2)$ with $E_j(t) \in C^\omega((-a_0^2, a_0^2), \mathbb{R})$;
- (2) $\exists C > 0$ s.t. $\|q_{Jz}\|_{\Sigma^r} \leq C|z|^3$, $|E_{Jz} - e_j| < C|z|^2$.

We set

$$\omega_{Jz} := \begin{cases} \sqrt{m^2 + E_{Jz}} & \text{for } J \leq n, \\ \sqrt{m^2 + E_{(J-n)z}} & \text{for } J > n \end{cases}, \quad \phi_J := \begin{cases} \phi_J & \text{for } J \leq n, \\ \phi_{J-n} & \text{for } J > n \end{cases}, \quad q_{Jz} := \begin{cases} q_{Jz} & \text{for } J \leq n, \\ q_{(J-n)z} & \text{for } J > n \end{cases}, \quad (1.6)$$

and further set $\tilde{\omega}_J$ as (1.4). Let $Q_{Jz} = z\phi_J + q_{Jz}$ for all $J \leq 2n$. Then $u(t, x) = e^{\pm it\omega_{Jz}} Q_{Jz}(x)$ are solutions to (??). For $z \in D_{\mathbb{C}^{2n}}(0, b_0)$ for $b_0 > 0$ sufficiently small we consider

$$\Phi_J[z] := (Q_{Jz}, i\tilde{\omega}_{Jz} Q_{Jz}) \text{ for } z \in \mathbb{C} \text{ and } \Phi[z] := \sum_{J=1}^{2n} \Phi_J[z_J] \text{ for } z \in \mathbb{C}^{2n}. \quad (1.7)$$

Notice that by the definition of Φ and Proposition 1.1, we have the gauge property $\Phi[e^{i\theta}z] = e^{i\theta}\Phi[z]$.

In $L^2(\mathbb{R}^3, \mathbb{C}) \times L^2(\mathbb{R}^3, \mathbb{C})$ we consider the symplectic form

$$\Omega(U|V) := 2 \langle \mathbb{J}^{-1}U|V \rangle, \text{ where } \mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (1.8)$$

Definition 1.2. Let $z = (z_1, \dots, z_{2n})$, $z_{JR} = \operatorname{Re} z_J$, $z_{JI} = \operatorname{Im} z_J$. For any $z \in D_{\mathbb{C}^{2n}}(0, b_0)$ with $b_0 > 0$ be sufficiently small and for $\partial_{JA} = \partial_{z_{JA}}$ we set

$$\mathcal{H}_c[z] := \{(\eta_1, \eta_2) \in L^2 \times L^2 : \Omega(\partial_{JA}\Phi[z])(\eta_1, \eta_2) = 0 \ \forall J \text{ and } A = R, I\}. \quad (1.9)$$

Remark 1.3. Notice that $\mathcal{H}_c[0] = L_c^2 \times L_c^2$ where $L_c^2 = P_c L^2$ is the continuous space associated to H in $L^2(\mathbb{R}^3, \mathbb{C})$. Here,

$$P_c u = u - \sum_{j=1}^n (\langle u, \phi_j \rangle \phi_j + \langle u, i\phi_j \rangle i\phi_j). \quad (1.10)$$

Remark 1.4. By $\Phi[e^{i\theta}z] = e^{i\theta}\Phi[z]$ we have $(\eta_1, \eta_2) \in \mathcal{H}_c[z] \Leftrightarrow (e^{i\theta}\eta_1, e^{i\theta}\eta_2) \in \mathcal{H}_c[e^{i\theta}z]$.

More generally, given a space $\Sigma^k(\mathbb{R}^3, \mathbb{C})$ we write $\Sigma_c^k = P_c \Sigma^k(\mathbb{R}^3, \mathbb{C})$. Notice that P_c can be defined on Σ^k even if k is negative.

Throughout the paper, we say that a pair (p, q) is *admissible* when

$$2/p + 3/q = 3/2, \quad 6 \geq q \geq 2, \quad p \geq 2. \quad (1.11)$$

The following theorem is our main result.

Theorem 1.5. Assume (H1)–(H4). Then there exist $\epsilon_0 > 0$ and $C > 0$ such that if we set $\epsilon := \|(u(0), v(0))\|_{H^1 \times L^2} < \epsilon_0$, then the solution $(u(t), v(t))$ of (1.1) can be written uniquely for all times and with $(\eta(t), \xi(t)) \in \mathcal{H}_c[z(t)]$ as

$$(u(t), v(t)) = \Phi[z(t)] + (\eta(t), \xi(t)). \quad (1.12)$$

such that there exist a unique J_0 , a $\rho_+ \in [0, \infty)^{2n}$ with $\rho_{+J} = 0$ for $J \neq J_0$, s.t. $|\rho_+| \leq C\epsilon$, and there exists (u_+, v_+) with $\|(u_+, v_+)\|_{H^1 \times L^2} \leq C\epsilon$ and such that, for $u_{lin}(u_+, v_+)(t) = K'_0(t)u_+ + K_0(t)v_+$ where $K_0(t) = \frac{\sin(t\sqrt{-\Delta+m^2})}{\sqrt{-\Delta+m^2}}$, we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \|(\eta(t), \xi(t)) - (u_{lin}(u_+, v_+)(t), \frac{d}{dt}u_{lin}(u_+, v_+)(t))\|_{H^1 \times L^2} &= 0, \\ \lim_{t \rightarrow +\infty} |z_J(t)| &= \rho_{+J}. \end{aligned} \quad (1.13)$$

Furthermore, we have $(\eta, \xi) = (\tilde{\eta}, \tilde{\xi}) + \mathbf{A}(t, x)$ s.t. for all admissible pairs (p, q)

$$\begin{aligned} \|z\|_{L_t^\infty(\mathbb{R}_+)} + \|(\tilde{\eta}, \tilde{\xi})\|_{L_t^p(\mathbb{R}_+, W_x^{\frac{1}{q}-\frac{1}{p}, q} \times W_x^{\frac{1}{q}-\frac{1}{p}-1, q})} &\leq C\epsilon, \\ \|\dot{z}_J + i\omega_J z_J\|_{L_t^\infty(\mathbb{R}_+)} &\leq C\epsilon^2, \end{aligned} \quad (1.14)$$

and s.t. $\mathbf{A}(t, \cdot) \in \Sigma^2 \times \Sigma^2$ for all $t \geq 0$, and

$$\lim_{t \rightarrow +\infty} \|\mathbf{A}(t, \cdot)\|_{\Sigma^2 \times \Sigma^2} = 0. \quad (1.15)$$

As we have already mentioned above, this theorem extends [28], which dealt with (u_0, v_0) both real valued, $n = 1$ in (H2), and with the further restriction that $3\omega_1 > m$. It extends also [2] in the sense that while [2] allows even more general spectra than (H2), [2] considers only (u_0, v_0) both real valued. The fact that (u_0, v_0) both are real valued. and the convergence is to 0 allows in [2] a simpler choice of coordinates systems. This is key because, like in [2, 9, 11], the main point in the proof of Theorem 1.5 consists in finding an appropriate system of coordinates where it is easier to extract an appropriate effective hamiltonian. Indeed, the purpose is to exploit the hamiltonian nature of the equation to prove damping through dispersion of most of the z_J 's. A simple example of this damping mechanism is given in the following, which we quote from [10]. Consider the hamiltonian

$$\mathcal{H}(z, h) = |z|^2 + \|\nabla h\|_{L^2}^2 + |z|^2 \bar{z} \int_{\mathbb{R}^2} G(x) h(x) dx + |z|^2 z \int_{\mathbb{R}^2} \bar{G}(x) \bar{h}(x) dx \quad (1.16)$$

for which the equilibrium $(0, 0)$ is asymptotically stable, as we sketch heuristically now. We have

$$i\dot{h} = -\Delta h + |z|^2 z \bar{G} \text{ and} \quad (1.17)$$

$$i\dot{z} = z + 2|z|^2 \int_{\mathbb{R}^2} h(x) G(x) dx + z^2 \int_{\mathbb{R}^2} \bar{h}(x) \bar{G}(x) dx. \quad (1.18)$$

If we set

$$h = -|z|^2 z R_{-\Delta}^+(1) \bar{G} + g \text{ for } R_{-\Delta}^+(1) = \lim_{\varepsilon \rightarrow 0^+} R_{-\Delta}(1 + i\varepsilon)$$

and we substitute in (1.18), ignoring g since it is smaller, we get

$$i\dot{z} = z - 2|z|^4 z \int_{\mathbb{R}^2} G R_{-\Delta}^+(1) \bar{G} dx - |z|^4 z \int_{\mathbb{R}^2} \bar{G} R_{-\Delta}^-(1) G dx.$$

Recall $R_{-\Delta}^\pm(\lambda) = P.V.(-\Delta - \lambda)^{-1} \pm i\pi\delta(-\Delta - \lambda)$ for any $\lambda > 0$. Multiplying by \bar{z} and taking imaginary part we get

$$\frac{d}{dt}|z|^2 = -2\pi\mathfrak{c}|z|^8 \text{ with } \mathfrak{c} = \int_{\mathbb{R}^2} G\delta(-\Delta - 1)\bar{G} dx \geq 0. \quad (1.19)$$

We conclude that $\mathfrak{c} \geq 0$ by the following formula, see ch.2 [13]:

$$\mathfrak{c} = \frac{1}{2} \int_{|\xi|=1} |\widehat{G}(\xi)|^2 d\sigma(\xi). \quad (1.20)$$

Assuming $\mathfrak{c} > 0$ (the meaning of hypothesis (H4) is basically this), which is generically true, then (1.19) yields the explicit formula

$$|z(t)|^2 = \frac{|z(0)|^2}{(1 + 6\pi\mathfrak{c}|z(0)|^2 t)^{\frac{1}{3}}}.$$

In the meantime, h scatters because one can apply Strichartz estimates to (1.17).

So the work in all the papers [2, 9, 11], as well as here, consists in finding a coordinate system where (1.1) has hamiltonian that, up to negligible error terms, is similar to (1.16). In [2] this is simpler because there the attractor set is formed just by the vacuum solution of (1.1). Here as well as in [9, 11] the attractor is more complex and leads to solutions with complicated trajectories around the attractor, as shown in [32], which treats for the NLS a similar problem but only for $n = 2$ in (H2).

As discussed in [9], the approach in [32] involving guesses on the trajectory of a solution, the turns of the solution away from unstable standing waves and, usually, its final convergence either to 0 or to a stable standing wave, appears a considerably difficult task under our hypothesis (H2), which is much more complex combinatorially than the situation in [32]. In this paper we frame the problem as in [9] and extend to the NLKG equation the NLS result obtained in [9], exactly as in [11] the result of [9] has been extended to Dirac equations. It turns out that the NLKG presents no significant new problems with respect to the NLS. In this sense this paper contains applications of ideas introduced already in [9]. However, due to the importance of the NLKG, we think the result in the present paper significant nonetheless. This in view of the fact that, apart from [28, 2] and [1], not much has been written about the asymptotic stability of standing waves of the NLKG. This in contrast to the rather extensive literature on asymptotic stability of standing waves of the NLS, e.g. [25, 26, 24], [4, 5], [30]–[33], [27], [14, 15, 16], [8, 7] and therein, and various other papers on the problem treated in [9] about the NLS, e.g. [18, 22, 21]. There are even some papers on asymptotic stability for the nonlinear Dirac equation, e.g. [23, 3, 11].

Now we give a quick description of the proof of Theorem 1.5. Following ideas from [18], subsequently elaborated in [9], we find a natural system of coordinates $(z_1, \dots, z_{2n}, \Xi)$ for (1.1), which comprise both discrete modes z_J for $J = 1, \dots, 2n$ and continuous modes Ξ . Since one of the discrete modes possibly does not decay, early in the paper a new auxiliary variable \mathbf{Z} is introduced. In the proof it is shown that all the components of \mathbf{Z} decay to 0. The components of \mathbf{Z} are the products $z_J \bar{z}_K$ with $J \neq K$. The role of the discrete mode z in (1.16) is taken by \mathbf{Z} in the hamiltonian of (1.1). Some elementary but essential lemmas about monomials in the variable \mathbf{Z} are then introduced in Sect. 2.3.

In the simplified setup of [2, 28] the initial coordinates are Darboux for the symplectic structure in the problem, but not here. Like in [7, 9] here we need instead to change coordinates to reduce to Darboux coordinates. Like in [9] all the coordinate changes in this paper satisfy

$$\begin{aligned} z'_1 &= z_1 + O(z\Xi) + O(\Xi^2) + \sum_{J \neq K} O(z_J z_K), \dots, z'_{2n} = z_{2n} + O(z\Xi) + O(\Xi^2) + \sum_{J \neq K} O(z_J z_K), \\ \Xi' &= \Xi + O(z\Xi) + O(\Xi^2) + \sum_{J \neq K} O(z_J z_K). \end{aligned} \tag{1.21}$$

We take an appropriate expansion of the energy in terms of these coordinates emphasizing (z, \mathbf{Z}, Ξ) . Like in [9], in the Darboux coordinates an important cancelation occurs in the energy, and specifically in the 2nd line of (5.1) the summations start from $l = 1$ and not from $l = 0$ like in the analogous expansion in (3.3). We provide a simplified explanation with respect to [9] for this cancelation in the course of Lemma 5.1. The hamiltonian is now of the same type of that in [9] so applying Theorem 5.9 [9] we obtain a hamiltonian somewhat similar to (1.16). There is a mechanism of nonlinear discrete continuous interaction similar to that sketched for (1.16) that yields the stabilization mechanism of Theorem 1.5, with $\mathbf{Z}(t) \xrightarrow{t \rightarrow \infty} 0$ and scattering of Ξ . Thanks to the structure of the coordinate changes (1.21) this behavior transfers from one coordinate system to the other and yields the decay of all the $z_J(t)$ except for at most one and the scattering of the continuous components.

For the scattering of the continuous modes we use a number of Strichartz and smoothing estimates stated in [2], in part proved there and in part gathered from the literature, mainly from [12], but see [2] for further references.

Finally, one thing that we do not accomplish here and which was proved instead in [9] is the instability of the excited states.

2 Notation, coordinates and resonant sets

2.1 Notation

- We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- We denote $z = (z_1, \dots, z_{2n})$, $|z| := \sqrt{\sum_{j=1}^{2n} |z_j|^2}$.
- Given a Banach space X , $v \in X$ and $\delta > 0$ we set $D_X(v, \delta) := \{x \in X \mid \|v - x\|_X < \delta\}$.
- Let A be an operator on $L^2(\mathbb{R}^3)$. Then $\sigma_p(A) \subset \mathbb{C}$ is the set of eigenvalues of A and $\sigma_e(A) \subset \mathbb{C}$ is the essential spectrum of A .
- For $m < 0$ and integer we set $\Sigma^m = (\Sigma^{-m})'$. Notice that the spaces Σ^r can be equivalently defined using for $r \in \mathbb{R}$ the norm $\|u\|_{\Sigma^r} := \|(1 - \Delta + |x|^2)^{\frac{r}{2}} u\|_{L^2}$.
- For $f : \mathbb{C}^{2n} \rightarrow \mathbb{C}$ we set $\partial_{JA} f(z) := \frac{\partial}{\partial z_{JA}} f(z)$, $\partial_J := \partial_{z_J}$ and $\partial_{\bar{J}} := \partial_{\bar{z}_J}$. Here as customary $\partial_{z_J} = \frac{1}{2}(\partial_{JR} - i\partial_{JI})$ and $\partial_{\bar{z}_J} = \frac{1}{2}(\partial_{JR} + i\partial_{JI})$.
- Occasionally we use a single index $\ell = J, \bar{J}$. To define $\bar{\ell}$ we use the convention $\bar{\bar{J}} = J$. We will also write $z_{\bar{J}} = \bar{z}_J$.
- We will consider vectors $z = (z_1, \dots, z_{2n}) \in \mathbb{C}^{2n}$ and for vectors $\mu, \nu \in (\mathbb{N} \cup \{0\})^{2n}$ we set $z^\mu \bar{z}^\nu := z_1^{\mu_1} \dots z_{2n}^{\mu_{2n}} \bar{z}_1^{\nu_1} \dots \bar{z}_{2n}^{\nu_{2n}}$. We will set $|\mu| = \sum_j \mu_j$.
- We have $dz_J = dz_{JR} + idz_{JI}$, $d\bar{z}_J = dz_{JR} - idz_{JI}$.
- Given two Banach spaces X and Y we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators $X \rightarrow Y$ with the norm of the uniform operator topology.
- We set $\mathbf{L}^2 := L^2 \times L^2$, $\Sigma^r := \Sigma^r \times \Sigma^r$ and $\mathbf{H}^s := H^s \times H^s$.
- $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

2.2 Coordinates

In the following, we fix $r_0 > 0$ sufficiently large.

A preliminary step for the choice of coordinates is the following standard ansatz.

Lemma 2.1. *There exist $c_0 > 0$, and $C > 0$ s.t. for all $(u, v) \in H^1 \times L^2$ with $\|(u, v)\|_{H^1 \times L^2} < c_0$, there exists a unique pair $(z, \tilde{\Xi}) \in \mathbb{C}^{2n} \times ((H^1 \times L^2) \cap \mathcal{H}_c[z])$ s.t.*

$$(u, v) = \Phi[z] + \tilde{\Xi} \quad (2.1)$$

with $|z| + \|\tilde{\Xi}\|_{H^1 \times L^2} \leq C\|(u, v)\|_{H^1 \times L^2}$.

The map $u \rightarrow (z, \tilde{\Xi})$ is $C^\omega(D_{H^1}(0, c_0), \mathbb{C}^{2n} \times H^1 \times L^2)$, and satisfies the gauge property:

$$z(e^{i\vartheta}u, e^{i\vartheta}v) = e^{i\vartheta}z(u, v), \text{ and } \Psi(e^{i\vartheta}u, e^{i\vartheta}v) = e^{i\vartheta}\Psi(u, v). \quad (2.2)$$

Proof. We consider for $J = 1, \dots, 2n$ and $A = R, I$ the functions

$$F_{JA}(u, v, z) := \Omega(\partial_{JA}\Phi[z]|(u, v) - \Phi[z]).$$

The F_{JA} is analytic in $\mathbf{L}^2 \times D_{\mathbb{C}^{2n}}(0, b_0)$ for the b_0 in Def. 1.2. We have

$$F_{JR}(0, 0, z) = -4\tilde{\omega}_J z_{JR} + O(z^3), \quad F_{JI}(0, 0, z) = -4\tilde{\omega}_{J0} z_{JI} + O(z^3). \quad (2.3)$$

The map $(u, v) \rightarrow z$ in $C^\omega(D_{\mathbf{L}^2}(0, c_0), \mathbb{C}^{2n})$ for a $c_0 > 0$ sufficiently small is obtained by implicit function theorem. All the other statements are equally elementary. For a proof in a similar set up see [9]. \square

We introduce

$$B := \sqrt{-\Delta + V + m^2}, \quad B^{-\frac{\sigma_3}{2}} = \begin{pmatrix} B^{-\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} \quad (2.4)$$

We need a system of independent coordinates, which the (z, Ψ) in (2.1) are not. The following lemma is used to complete (later in Lemma 2.3) the z with a continuous coordinate.

Lemma 2.2. *There exists $d_0 > 0$ such that there exist $C_{JA} \in C^\omega(D_{\mathbb{C}^{2n}}(0, d_0), \Sigma^{r_0})$ for $J = 1, \dots, 2n$ and $A = R, I$ such that*

(i) *For $R[z]$ defined by*

$$R[z]\Xi = B^{-\frac{\sigma_3}{2}}\Xi + \sum_{J'=1}^{2n} \sum_{A'=R,I} \langle C_{J'A'}[z]|\Xi \rangle \partial_{J'A'}\Phi[0], \quad (2.5)$$

we have $R[z] : \mathbf{H}_c^{\frac{1}{2}} \rightarrow (H^1 \times L^2) \cap \mathcal{H}_c[z]$ and

$$B^{\frac{\sigma_3}{2}} \circ (P_c \times P_c) \Big|_{\mathcal{H}_c[z]} = R[z]^{-1}. \quad (2.6)$$

(ii) $\|C_{JA}[z]\|_{\Sigma^r} \leq c_r |z|^2$ for all $r \geq 0$.

(iii) $R[e^{i\theta}z] = e^{i\theta}R[z]e^{-i\theta}$ for all $\theta \in \mathbb{R}$.

Proof. We define $C'_{JA}[z]\Xi \in \mathbb{R}$ ($J = 1, \dots, 2n, A = R, I$) to be the unique solution of the system

$$\Omega \left(\partial_{JA}\Phi[z] \Big| B^{-\frac{\sigma_3}{2}}\Xi + \sum_{J'=1}^{2n} \sum_{A'=R,I} (C'_{J'A'}[z]\Xi) \partial_{J'A'}\Phi[0] \right) = 0, \quad J = 1, \dots, 2n, A = R, I, \quad (2.7)$$

where $\partial_{JR}\Phi[0] = (\phi_J, i\tilde{\omega}_J\phi_J)$ and $\partial_{JI}\Phi[0] = i(\phi_J, i\tilde{\omega}_J\phi_J) = i\partial_{JR}\Phi[0]$. Rewriting (2.7), we have

$$\sum_{J'=1}^{2n} \sum_{A'=R,I} \Omega(\partial_{JA}\Phi[z]|\partial_{J'A'}\Phi[0])C'_{J'A'}[z]\Xi = -\Omega(\partial_{JA}\Phi[z]|B^{-\frac{\sigma_3}{2}}\Xi) \quad (2.8)$$

Since the coefficient matrix is invertible for z sufficiently small, we can solve (2.7). Furthermore, from (2.8), we see that $C'_{J'A'}[z]\Xi$ can be written as $\langle C_{JA}[z]|\Xi \rangle$ for some $C_{JA}[z] \in \Sigma^r$ for arbitrary

r . The real analyticity with respect to z follows from the real analyticity of $\Phi[z]$. Notice that since $|\Omega(\partial_{J'A'}\Phi[z]|B^{-\frac{\sigma_3}{2}}\Xi)| \lesssim |z|^2$, we have (ii).

Next we define $R[z]$ by (2.5) using these $C_{JA}[z]$. From (2.7) we obtain $R[z] : \mathbf{H}_c^{\frac{1}{2}} \rightarrow (H^1 \times L^2) \cap \mathcal{H}_c[z]$. Furthermore, (2.6) follows from the form of $R[z]$ and the uniqueness of the solution of (2.7).

We will postpone (iii) to the appendix of this paper. \square

By $R[z]$ given in Lemma 2.2, we have a system of independent coordinates which we needed.

Lemma 2.3. *For the $d_0 > 0$ of Lemma 2.2 the map $\mathcal{F} : (z, \eta, \xi) \mapsto U := (u, v)$ defined for $\Xi = (\eta, \xi)$ by*

$$\mathcal{F}(z, \Xi) = \Phi[z] + R[z]\Xi \text{ for } (z, \Xi) \in D_{\mathbb{C}^{2n}}(0, d_0) \times \mathbf{H}_c^{\frac{1}{2}} \quad (2.9)$$

is with values in $H^1 \times L^2$ and is C^ω . Furthermore, there is a $d_1 > 0$ such that for $(z, \Xi) \in B_{\mathbb{C}^{2n}}(0, d_1) \times B_{\mathbf{H}_c^{\frac{1}{2}}}(0, d_1)$ the above map is a diffeomorphism and

$$|z| + \|\Xi\|_{\mathbf{H}_c^{\frac{1}{2}}} \sim \|\mathcal{F}(z, \Xi)\|_{H^1 \times L^2}. \quad (2.10)$$

Finally, setting $\Psi[z, \Xi] := \Phi[z] + (R[z] - B^{-\frac{\sigma_3}{2}})\Xi$, we have $\Psi \in C^\omega(D_{\mathbb{C}^{2n}} \times \Sigma_c^{-r_0}, \Sigma^{r_0})$.

2.3 Some simple combinatorics

We now recall from [9] the following definition, where we are introducing the auxiliary quantity \mathbf{Z} .

Definition 2.4. Given $z \in \mathbb{C}^n$, we denote by \mathbf{Z} the vector (or the matrix) with entries $(z_J \bar{z}_K)$ with $J, K \in [1, 2n]$ but only with pairs of indexes with $J \neq K$. Here $\mathbf{Z} \in L$, where L is the subspace of $\mathbb{C}^{n_0} = \{(a_{J,K})_{J,K=1,\dots,n} : J \neq K\}$ for $n_0 = 2n(2n-1)$, with $(a_{J,K}) \in L$ iff $a_{J,K} = \bar{a}_{K,J}$ for all J, K . For a multi index $\mathbf{m} = \{m_{JK} \in \mathbb{N}_0 : J \neq K\}$ we set $\mathbf{Z}^{\mathbf{m}} = \prod (z_J \bar{z}_K)^{m_{JK}}$ and $|\mathbf{m}| := \sum_{J,K} m_{JK}$.

We will think formally of \mathbf{Z} as an auxiliary variable and we will be dealing with polynomials in the variable \mathbf{Z} . Some of the monomials in \mathbf{Z} will be reminder terms. In order to distinguish between the monomials which are reminder terms and the ones which aren't, we need the following definition.

Definition 2.5. Consider the set of multi indexes \mathbf{m} as in Definition 2.4. Consider for any $K \in \{1, \dots, 2n\}$ the set

$$\begin{aligned} \mathcal{M}_K(r) &= \{\mathbf{m} : \left| \sum_{L=1}^{2n} \sum_{J=1}^{2n} m_{LJ}(\tilde{\omega}_L - \tilde{\omega}_J) - \tilde{\omega}_K \right| > r \text{ and } |\mathbf{m}| \leq r\}, \\ \mathcal{M}_0(r) &= \{\mathbf{m} : \sum_{L=1}^{2n} \sum_{J=1}^{2n} m_{LJ}(\tilde{\omega}_L - \tilde{\omega}_J) = 0 \text{ and } |\mathbf{m}| \leq r\}. \end{aligned} \quad (2.11)$$

Set now

$$\begin{aligned} M_K(r) &= \{(\mu, \nu) \in \mathbb{N}_0^{2n} \times \mathbb{N}_0^{2n} : \exists \mathbf{m} \in \mathcal{M}_K(r) \text{ such that } z^\mu \bar{z}^\nu = \bar{z}_K \mathbf{Z}^{\mathbf{m}}\}, \\ M(r) &= \cup_{K=1}^{2n} M_K(r). \end{aligned} \quad (2.12)$$

We also set $M = M(2N+4)$, and

$$M_{\min} = \{(\mu, \nu) \in M : (\alpha, \beta) \in M \text{ with } \alpha_J \leq \mu_J \text{ and } \beta_J \leq \nu_J \forall J \Rightarrow (\alpha, \beta) = (\mu, \nu)\}. \quad (2.13)$$

It is easy to see that if $(\mu, \nu) \in M_{min}$, then for any J we have $\mu_J \nu_J = 0$. Indeed, first of all $(\mu, \nu) \in M(r)$ if and only if $|\nu| = |\mu| + 1$, $|\mu| \leq r$ and $|(\mu - \nu) \cdot \tilde{\omega}| > m$. Now, if $\mu_{J_0} \geq 1$ and $\nu_{J_0} \geq 1$ then, by subtracting from both of them a unit and leaving unchanged the other coordinates, we obtain another pair $(\alpha, \beta) \in M_{min}$ with $\alpha_J \leq \mu_J$ and $\beta_J \leq \nu_J$ for all J but with $(\alpha, \beta) \neq (\mu, \nu)$, so that $(\mu, \nu) \notin M_{min}$.

Lemma 2.6. *We have the following facts.*

- (1) Let $\mathbf{a} = (a_{LJ}) \in \mathbb{N}_0^{n_0}$ s.t. for the N in (H3)

$$\sum_{L < J \leq n} a_{LJ} + \sum_{L \leq n < J} a_{JL} + \sum_{n < L < J} a_{JL} > N. \quad (2.14)$$

Then for any K , we have (notice the switch in the indexes)

$$\sum_{L < J \leq n} a_{LJ}(\omega_L - \omega_J) - \sum_{L \leq n < J} a_{JL}(\omega_L + \omega_J) + \sum_{n < L < J} a_{JL}(\omega_L - \omega_J) + \omega_K < -m. \quad (2.15)$$

- (2) Consider $\mathbf{m} \in \mathbb{N}_0^{n_0}$ with $|\mathbf{m}| \geq 2N + 3$ and the monomial $z_{J_0} \mathbf{Z}^{\mathbf{m}}$. Then $\exists \mathbf{a}, \mathbf{b} \in \mathbb{N}_0^{n_0}$ s.t. for both $\mathbf{f} = \mathbf{a}, \mathbf{b}$

$$\begin{aligned} \sum_{L < J \leq n} f_{LJ} + \sum_{L \leq n < J} f_{JL} + \sum_{n < L < J} f_{JL} &= N + 1, \\ f_{LJ} &= 0 \text{ for all } (L, J) \text{ not in the above sum, and } a_{LJ} + b_{LJ} \leq m_{LJ} + m_{JL} \text{ for all } (L, J). \end{aligned} \quad (2.16)$$

Moreover, there are two indexes (K, S) s.t.

$$\sum_{L, J} a_{LJ}(\tilde{\omega}_L - \tilde{\omega}_J) - \tilde{\omega}_K < -m \text{ and } \sum_{L, J} b_{LJ}(\tilde{\omega}_L - \tilde{\omega}_J) - \tilde{\omega}_S < -m, \quad (2.17)$$

and for $|z| \leq 1$

$$|z_{J_0} \mathbf{Z}^{\mathbf{m}}| \leq |z_{J_0}| |z_K \mathbf{Z}^{\mathbf{a}}| |z_S \mathbf{Z}^{\mathbf{b}}|. \quad (2.18)$$

- (3) For \mathbf{m} with $|\mathbf{m}| \geq 2N + 3$, there exist (K, S) , $\mathbf{a} \in \mathcal{M}_K$ and $\mathbf{b} \in \mathcal{M}_S$ s.t. (2.18) holds.

Proof. The inequality (2.15) follows immediately by the definition of N and

$$\sum_{L < J} a_{LJ}(\omega_L - \omega_J) + \omega_K < -\min\{\omega_J - \omega_L : J > L\}N + m \leq -m.$$

We now consider the monomial $z_{J_0} \mathbf{Z}^{\mathbf{m}}$. Since $|\mathbf{m}| \geq 2N + 3$, there are vectors $\mathbf{c}, \mathbf{d} \in \mathbb{N}_0^{2n_0}$ such that $|\mathbf{c}| = |\mathbf{d}| = N + 1$ with $c_{LJ} + d_{LJ} \leq m_{LJ}$ for all (L, J) .

Focusing on \mathbf{c} we define \mathbf{a} as follows:

$$\begin{aligned} \text{if } L < J \leq n & \text{ let } a_{LJ} = c_{LJ} + c_{JL} \text{ and } a_{JL} = 0; \\ \text{if } L \leq n < J & \text{ let } a_{LJ} = 0 \text{ and } a_{JL} = c_{LJ} + c_{JL}; \\ \text{if } n < L < J & \text{ let } a_{LJ} = 0 \text{ and } a_{JL} = c_{LJ} + c_{JL}. \end{aligned}$$

Notice that

$$\tilde{\omega}_L - \tilde{\omega}_J = \begin{cases} \omega_L - \omega_J & \text{if } L < J \leq n \\ \omega_L + \omega_J & \text{if } L \leq n < J \\ -\omega_L + \omega_J & \text{if } n < L < J \end{cases}$$

and that

$$N + 1 = \sum_{L,J} a_{L,J} = \sum_{L < J \leq n} a_{L,J} + \sum_{L \leq n < J} a_{J,L} + \sum_{n < L < J} a_{J,L}.$$

Hence, by claim (1)

$$\begin{aligned} & \sum_{L,J} a_{L,J}(\tilde{\omega}_L - \tilde{\omega}_J) - \tilde{\omega}_K \\ &= \sum_{L < J \leq n} a_{L,J} \overbrace{(\tilde{\omega}_L - \tilde{\omega}_J)}^{\omega_L - \omega_J} + \sum_{L \leq n < J} a_{J,L} \overbrace{(\tilde{\omega}_J - \tilde{\omega}_L)}^{-(\omega_L + \omega_J)} + \sum_{n < L < J} a_{J,L} \overbrace{(\tilde{\omega}_J - \tilde{\omega}_L)}^{\omega_L - \omega_J} - \tilde{\omega}_K \\ &\leq \sum_{L < J \leq n} a_{L,J}(\omega_L - \omega_J) - \sum_{L \leq n < J} a_{J,L}(\omega_L + \omega_J) + \sum_{n < L < J} a_{J,L}(\omega_L - \omega_J) + \omega_K < -m. \end{aligned}$$

We can define \mathbf{b} from \mathbf{d} similarly. Then, we have

$$z_{J_0} \mathbf{Z}^{\mathbf{m}} = z_{J_0} z^\mu \bar{z}^\nu \mathbf{Z}^{\mathbf{c}} \mathbf{Z}^{\mathbf{d}} \text{ with } |\mu| > 0, \text{ and } |\nu| > 0. \quad (2.19)$$

Therefore, for z_K a factor of z^μ and \bar{z}_S a factor of \bar{z}^ν , for $|z| \leq 1$ we have from (2.19)

$$|z_{J_0} \mathbf{Z}^{\mathbf{m}}| \leq |z_{J_0}| |z_K \mathbf{Z}^{\mathbf{c}}| |z_S \mathbf{Z}^{\mathbf{d}}| = |z_{J_0}| |z_K \mathbf{Z}^{\mathbf{a}}| |z_S \mathbf{Z}^{\mathbf{b}}|.$$

Furthermore, (2.16) is satisfied. Moreover, since our (\mathbf{a}, \mathbf{b}) satisfy $\mathbf{a} \in \mathcal{M}_K$ and $\mathbf{b} \in \mathcal{M}_L$, claim (3) is a consequence of claim (2). \square

3 The energy functional

For $U = (u, v)$, the equation (1.1) admits the following energy

$$\begin{aligned} E(U) &= \langle \mathcal{A}U | U \rangle + E_P(u), \\ \text{where } \mathcal{A} &= \begin{pmatrix} B^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } E_P(u) := 2^{-1} \int_{\mathbb{R}^3} |u|^4 dx. \end{aligned} \quad (3.1)$$

We have $E \in C^\omega(H^1(\mathbb{R}^3, \mathbb{C}) \times L^2(\mathbb{R}^3, \mathbb{C}), \mathbb{R})$. We denote by dE the Frechét derivative of E . We also define ∇E by

$$dE(U)X = \langle \nabla E(U) | X \rangle. \quad (3.2)$$

Similarly, for $F = F(z, \Xi)$, we set $\nabla_\Xi F$ by $d_\Xi F(z, \Xi) \tilde{\Xi} = \left\langle \nabla_\Xi F(z, \Xi), \tilde{\Xi} \right\rangle$, where $d_\Xi F$ is the Frechét derivative with respect to Ξ .

We set $E^0(z, \Xi) := \mathcal{F}^* E(z, \Xi)$ where \mathcal{F} is given in Lemma 2.3.

Lemma 3.1. For $(z, \Xi) \in D_{\mathbb{C}^{2n}}(0, d_1) \times (D_{\mathbf{H}^{\frac{1}{2}}}(0, d_1) \cap \mathcal{H}_c[0])$ we have

$$\begin{aligned}
E^0(z, \Xi) &= \sum_{J=1}^{2n} E(\Phi_J[z_J]) + \langle B\Xi|\Xi \rangle + E_P(B^{-\frac{1}{2}}\eta) \\
&+ \sum_{l=0}^{\infty} \sum_{J=0}^{2n} \sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} a_{J\mathbf{m}}^{(0)}(|z_J|^2) + \sum_{l=0}^{\infty} \sum_{J=1}^{2n} \sum_{|\mathbf{m}|=l} \langle z_J \mathbf{Z}^{\mathbf{m}} G_{J\mathbf{m}}^{(0)}(|z_J|^2) |\Xi \rangle \\
&+ \sum_{i+j=2} \langle G_{2ij}^{(0)}(z, \Xi) | (B^{-\frac{1}{2}}\eta)^i \overline{B^{-\frac{1}{2}}\eta}^j \rangle + \sum_{i+j=3} \langle G_{3ij}^{(0)}(z, \Xi) | B^{-\frac{1}{2}}\eta^i \overline{B^{-\frac{1}{2}}\eta}^j \rangle + \mathcal{R}^{(0)},
\end{aligned} \tag{3.3}$$

where $a_{J\mathbf{m}}^{(0)}(|z_0|^2)$ will mean $a_{J\mathbf{m}}^{(0)}$ (a constant), $\langle B\Xi|\Xi \rangle := \langle B\xi|\xi \rangle + \langle B\eta|\eta \rangle$ and:

- (1) $(a_{j\mathbf{m}}^{(0)}, G_{j\mathbf{m}}^{(0)}) \in C^\omega(D_{\mathbb{R}}(0, d_0^2), \mathbb{C} \times \Sigma^{r_0}(\mathbb{R}^3, \mathbb{C}))$ with $|a_{j\mathbf{m}}^{(0)}(|z_J|)| \leq c|z_J|^2$ and $G_{j0}^{(0)}(0) = 0$;
- (2) $(G_{2mij}^{(0)}, G_{dij}^{(0)}, \mathcal{R}^{(0)}) \in C^\omega(D_{\mathbb{C}^{2n} \times \Sigma^{-r_0}}(0, d_0), \Sigma^{r_0}(\mathbb{R}^3, \mathbb{C}) \times \Sigma^{r_0}(\mathbb{R}^3, \mathbb{C}) \times \Sigma^{r_0}(\mathbb{R}^3, \mathbb{C}))$;
- (3) $G_{2ij}^{(0)}(0) = 0$ and $|\mathcal{R}^{(0)}| \leq c|z| \|\Xi\|_{\Sigma^{-r_0}}^2$.

Proof. First, by Lemma 2.3, we have $\mathcal{F}(z, \Xi) = \Psi[z, \Xi] + B^{-\frac{\sigma_3}{2}}\Xi$. By Taylor expansion, we have

$$\begin{aligned}
E(U) &= E(\Psi[z, \Xi]) + \left\langle \nabla E(\Psi[z, \Xi]) | B^{-\frac{\sigma_3}{2}}\Xi \right\rangle \\
&+ \int_0^1 (1-t) \left\langle \nabla^2 E(\Psi[z, \Xi] + tB^{-\frac{\sigma_3}{2}}\Xi) B^{-\frac{\sigma_3}{2}}\Xi \mid B^{-\frac{\sigma_3}{2}}\Xi \right\rangle dt.
\end{aligned} \tag{3.4}$$

Since $\Psi \in C^\omega(D_{\mathbb{C}^{2n}} \times \Sigma_c^{-r}, \Sigma^r)$ for arbitrary r , we can expand the 1st term and the 2nd term as

$$\sum_{l=0}^{\infty} \sum_{J=0}^{2n} \sum_{|\mathbf{m}|=l} \mathbf{Z}^{\mathbf{m}} a_{J\mathbf{m}}^{(0)}(|z_J|^2) + \sum_{l=0}^{\infty} \sum_{J=1}^{2n} \sum_{|\mathbf{m}|=l} \langle z_J \mathbf{Z}^{\mathbf{m}} G_{J\mathbf{m}}^{(0)}(|z_J|^2) |\Xi \rangle + \mathcal{R}^{(0)}(z, \Xi), \tag{3.5}$$

where \mathcal{R} satisfies the estimate in (3) and $a_{0\mathbf{m}}^{(0)}(|z_0|^2) = a_{0\mathbf{m}}^{(0)}$ (constant). Notice that the pure term of z_J (i.e. a_{j0}) corresponds to $E(Q_{Jz_J}, i\tilde{\omega}_{Jz_J} Q_{Jz_J})$ and, since there are no constant terms, $a_{00}^{(0)} = 0$. We now consider the 3rd term of (3.4). For $\Psi(z, \Xi) = (\psi_1(z, \Xi), \psi_2(z, \Xi))$ and recalling $\Xi = (\eta, \xi)$,

$$\nabla^2 E(\Psi[z, \Xi] + tB^{-\frac{\sigma_3}{2}}\Xi) = 2 \begin{pmatrix} B^2 & 0 \\ 0 & 1 \end{pmatrix} + \nabla^2 E_P(\psi_1(z, \Xi) + B^{-1/2}\eta). \tag{3.6}$$

The contribution of the 1st term of the r.h.s. of (3.6) in the 3rd term of (3.4) is

$$\int_0^1 (1-t) \left\langle 2 \begin{pmatrix} B^2 & 0 \\ 0 & 1 \end{pmatrix} B^{-\frac{\sigma_3}{2}}\Xi \mid B^{-\frac{\sigma_3}{2}}\Xi \right\rangle = \langle B\Xi|\Xi \rangle. \tag{3.7}$$

Recall that $E_p(u) = \frac{1}{2} \int |u|^4 dx$. Elementary computations show that $\nabla E_p \in C^\omega(H^1, H^{-1})$ and $\nabla^2 E_p \in C^\omega(H^1, \mathcal{L}(H^1, H^{-1}))$ are defined by

$$\nabla E_p(u) = 2|u|^2 u, \quad \nabla^2 E_p(u)X = 2|u|^2 X + 4u \operatorname{Re}(u\overline{X}),$$

so that $\langle \nabla^2 E_p(u)X | X \rangle = 4(|u|^2 |X|^2) + 2\langle u^2 | X^2 \rangle$. Then, after elementary computations, the contribution of the 2nd term of the r.h.s. of (3.6) in the 3rd term of (3.4) is

$$2 \left\langle |\psi_1|^2 \mid B^{-1/2}\eta^2 \right\rangle + \left\langle \psi_1^2 \mid (B^{-1/2}\eta)^2 \right\rangle + 2 \left\langle \psi_1 \mid B^{-1/2}\eta^2 B^{-1/2}\eta \right\rangle + E_p(B^{-1/2}\eta). \tag{3.8}$$

Therefore, combining (3.4)–(3.8), we have the conclusion. \square

4 Darboux Theorem

We define Ω_0 by

$$\Omega_0(X|Y) = \sum_{J=1}^{2n} \Omega(d\Phi_J[z_J]X|d\Phi_J[z_J]Y) + \Omega(d\Xi|d\Xi). \quad (4.1)$$

Proposition 4.1 (Darboux theorem). *There exists $d_D > 0$ such that there exists $\tilde{\mathfrak{F}}^D = (\tilde{\mathfrak{F}}_z^D, \tilde{\mathfrak{F}}_\Xi^D) \in C^\omega(D_{\mathbb{C}^{2n} \times \Sigma^{-r_0}}(0, d_1), \mathbb{C}^{2n} \times \Sigma^{r_0})$ such that*

$$|\tilde{\mathfrak{F}}_z^D(z, \Xi)| + \|\tilde{\mathfrak{F}}_\Xi^D(z, \Xi)\|_{\Sigma^{r_0}} \leq c|z|(|\mathbf{Z}| + \|\Xi\|_{\Sigma^{-r_0}}), \quad (4.2)$$

and $\mathfrak{F}^D := \text{Id} + \tilde{\mathfrak{F}}^D$ satisfies $(\mathfrak{F}^D)^* \Omega = \Omega_0$. In addition, we have $\mathfrak{F}^D(e^{i\theta}z, e^{i\theta}\Xi) = e^{i\theta}\mathfrak{F}^D(z, \Xi)$.

The rest of this chapter is devoted for the proof of Proposition 4.1. It consists of three steps.

1. Find an appropriate 1-form Γ such that $\Omega - \Omega_0 = d\Gamma$.
2. For each $s \in [0, 1]$ find the vector field \mathcal{X}^s s.t. $i_{\mathcal{X}^s}\Omega_s = -\Gamma$, where $\Omega_s := (\Omega + s(\Omega - \Omega_0))$.
3. Solve $\frac{d}{ds}\mathfrak{F}^s = \mathcal{X}^s(\mathfrak{F}^s)$ with $\mathfrak{F}^0 = \text{Id}$.

Then, $\mathfrak{F} := \mathfrak{F}^1$ will be our desired transform. This will be seen by

$$\frac{d}{ds}(\mathfrak{F}^s)^*\Omega_s = (\mathfrak{F}^s)^*(\mathcal{L}_{\mathcal{X}^s}\Omega_s + \partial_s\Omega_s) = (\mathfrak{F}^s)^*(di_{\mathcal{X}^s}\Omega_s + d\Gamma) = 0. \quad (4.3)$$

The estimate (4.2) will be deduced from the construction.

It is elementary that $d\gamma = \Omega$ and $d\gamma_0 = \Omega_0$ for the following 1-forms:

$$\gamma(U)X := 2^{-1}\Omega(U|X), \quad \gamma_0(U)X := 2^{-1}\sum_{J=1}^{2n} \Omega(\Phi_J[z_J]|d\Phi_J[z_J]X) + 2^{-1}\Omega(\Xi|d\Xi). \quad (4.4)$$

we have .

Lemma 4.2. *For the $d_0 > 0$ of Lemma 2.3 set $D = D_{\mathbb{C}^{2n} \times \Sigma^{-r_0}}(0, d_0)$. Then there exist functions $\Gamma_{JA} \in C^\omega(D, \mathbb{R})$ and $\Gamma_\Xi \in C^\omega(D, \Sigma^{r_0})$ s.t. $\psi \in C^\omega(D, \mathbb{R})$ such that*

$$\gamma(U) - \gamma_0(U) - d\psi = \sum_{J=1}^{2n} \sum_{A=R,I} \Gamma_{JA} dz_{JA} + \langle \Gamma_\Xi | d\Xi \rangle := \Gamma. \quad (4.5)$$

Further, we have $\Gamma(e^{i\theta}U) = \Gamma(U)$ for all $\theta \in \mathbb{R}$ and

$$\sum_{J=1}^{2n} \sum_{A=R,I} |\Gamma_{JA}| + \|\Gamma_\Xi\|_{\Sigma^{r_0}} \leq c|z|(|\mathbf{Z}| + \|\Xi\|_{\Sigma^{-r_0}}), \quad (4.6)$$

Proof. Notice that, for $\tilde{R}[z] = R[z] - B^{-\frac{\sigma_3}{2}}$, expressing U and $X = dUX$ in terms of (2.9) we have

$$2\gamma(U)X = \Omega(U|X) = \Omega\left(\sum_{J=1}^{2n} \Phi_J[z_J] + B^{-\frac{\sigma_3}{2}}\Xi + \tilde{R}[z]\Xi \mid d\left(\sum_{K=1}^{2n} \Phi_K[z_K] + B^{-\frac{\sigma_3}{2}}\Xi + \tilde{R}[z]\Xi\right)X\right).$$

Therefore, using also $\mathbb{J}^{-1}\sigma_3 = -\sigma_3\mathbb{J}^{-1}$, which implies $B^{-\sigma_3/2}\mathbb{J}^{-1}B^{-\sigma_3/2} = \mathbb{J}^{-1}$, we have

$$\begin{aligned}
2(\gamma - \gamma_0) &= \sum_{J=1}^{2n} \Omega(\Phi_J[z_J] \mid \sum_{K \neq J} d\Phi_K[z_K] + B^{-\frac{\sigma_3}{2}} d\Xi + d(\tilde{R}[z]\Xi)) \\
&\quad + \Omega(B^{-\frac{\sigma_3}{2}} \Xi \mid \sum_{K=1}^{2n} d\Phi_K[z_K] + d(\tilde{R}[z]\Xi)) \\
&\quad + \Omega(\tilde{R}[z]\Xi \mid d(\sum_{K=1}^{2n} \Phi_K[z_K] + B^{-\frac{\sigma_3}{2}} \Xi + \tilde{R}[z]\Xi)).
\end{aligned} \tag{4.7}$$

Notice $2(\gamma - \gamma_0) - \sum_{J=1}^{2n} \Omega(\Phi_J[z_J] \mid B^{-\frac{\sigma_3}{2}} d\Xi)$ admits an expansion like the middle part of (4.5), where the coefficients satisfy (4.6). If we write

$$\sum_{J=1}^{2n} \Omega(\Phi_J[z_J] \mid B^{-\frac{\sigma_3}{2}} d\Xi) = d \sum_{J=1}^{2n} \Omega(\Phi_J[z_J] \mid B^{-\frac{\sigma_3}{2}} \Xi) + \Omega(B^{-\frac{\sigma_3}{2}} \Xi \mid d\Phi_J[z_J])$$

for $\psi = \sum_{J=1}^{2n} \Omega(\Phi_J[z_J] \mid B^{-\frac{\sigma_3}{2}} \Xi)$ we obtain (4.5)–(4.6). \square

Lemma 4.3. *Consider the form $\Omega_s := \Omega_0 + s(\Omega - \Omega_0)$ and set $i_X \Omega_t(Y) := \Omega_t(X|Y)$. Then $(\Omega - \Omega_0)$ and Γ extend to forms defined in $D_{\mathbb{C}^{2n}}(0, d_0) \times \Sigma_c^{-r_0}$ and there is a $d_1 \in (0, d_0)$ s.t. for any $(s, z, \Xi) \in (-4, 4) \times D_{\mathbb{C}^{2n} \times \Sigma_c^{-r_0}}(0, d_1)$ there exists exactly one solution $\mathcal{X}^s(z, \Xi) \in \mathbf{L}^2$ of the equation $i_{\mathcal{X}^s} \Omega_s = -\Gamma$. Furthermore, we have the following facts.*

- (i) $\mathcal{X}^s(z, \Xi) \in \Sigma^{r_0}$ and if we set $\mathcal{X}_{JA}^s(z, \Xi) = dz_{JA} \mathcal{X}^s(z, \Xi)$ and $\mathcal{X}_{\Xi}^s(z, \Xi) = d\Xi \mathcal{X}^s(z, \Xi)$, we have $\mathcal{X}_{JA}^s(z, \Xi) \in C^\omega(D_{\mathbb{C}^{2n} \times \Sigma^{-r_0}}(0, d_1) \times (-4, 4), \mathbb{R})$, $\mathcal{X}_{JA}^s(z, \Xi) \in C^\omega(D_{\mathbb{C}^{2n} \times \Sigma^{-r_0}}(0, d_1) \times (-4, 4), \Sigma_c^{r_0})$ and

$$\sum_{J=1}^{2n} \sum_{A=R, I} |\mathcal{X}_{JA}^s(z, \Xi)| + \|\mathcal{X}_{\Xi}^s(z, \Xi)\|_{\Sigma^{r_0}} \leq c|z|(|\mathbf{Z}| + \|\Xi\|_{\Sigma^{-r_0}}). \tag{4.8}$$

- (ii) For $\mathcal{X}_J^s := dz_J \mathcal{X}^s$ and $\mathcal{X}_{\Xi}^s = d\Xi \mathcal{X}^s$, we have $\mathcal{X}_J^s(e^{i\theta} z, e^{i\theta} \Xi) = e^{i\theta} \mathcal{X}_J^s(z, \Xi)$ and $\mathcal{X}_{\Xi}^s(e^{i\theta} z, e^{i\theta} \Xi) = e^{i\theta} \mathcal{X}_{\Xi}^s(z, \Xi)$.

Proof. In the following, we omit the summation. We directly solve $i_{\mathcal{X}^s} \Omega_t = -\Gamma$. First,

$$\Omega_0(\mathcal{X}^s|Y) = \Omega(\partial_{JB} \Phi_J[z_J] \mid \partial_{JA} \Phi_J[z_J]) \mathcal{X}_{JB}^s Y_{JA} + \Omega(\mathcal{X}_{\Xi}^s|Y_{\Xi}).$$

Next, since $\Omega - \Omega_0 = d\Gamma$, we have

$$\begin{aligned}
\Omega(\mathcal{X}^s|Y) - \Omega_0(\mathcal{X}^s|Y) &= (\partial_{KB} \Gamma_{JA} - \partial_{JA} \Gamma_{KB}) \mathcal{X}_{KB}^s Y_{JA} + \langle \nabla_{\Xi} \Gamma_{JA} | \mathcal{X}_{\Xi}^s \rangle Y_{JA} - \langle \nabla_{\Xi} \Gamma_{KB} | Y_{\Xi} \rangle \mathcal{X}_{KB}^s \\
&\quad + \langle \partial_{KB} \Gamma_{\Xi} | Y_{\Xi} \rangle \mathcal{X}_{KB}^s - \langle \partial_{JA} \Gamma_{\Xi} | \mathcal{X}_{\Xi}^s \rangle Y_{JA} + \langle d_{\Xi} \Gamma_{\Xi}(\mathcal{X}_{\Xi}^s) | Y_{\Xi} \rangle - \langle d_{\Xi} \Gamma_{\Xi}(Y_{\Xi}) | \mathcal{X}_{\Xi}^s \rangle.
\end{aligned}$$

Therefore, we have

$$2\mathbb{J}^{-1} \mathcal{X}_{\Xi}^s + s(-\mathcal{X}_{KB}^s \nabla_{\Xi} \Gamma_{KB} + \mathcal{X}_{KB}^s \partial_{KB} \Gamma_{\Xi} + d_{\Xi} \Gamma_{\Xi}(\mathcal{X}_{\Xi}^s) - (d_{\Xi} \Gamma_{\Xi})^* \mathcal{X}_{\Xi}^s) = -\Gamma_{\Xi}, \tag{4.9}$$

$$\begin{aligned}
\Omega(\partial_{JB} \Phi_J[z_J] \mid \partial_{JA} \Phi_J[z_J]) \mathcal{X}_{JB}^s + s((\partial_{KB} \Gamma_{JA} - \partial_{JA} \Gamma_{KB}) \mathcal{X}_{KB}^s + \langle \nabla_{\Xi} \Gamma_{JA} | \mathcal{X}_{\Xi}^s \rangle - \langle \partial_{JA} \Gamma_{\Xi} | \mathcal{X}_{\Xi}^s \rangle) \\
= -\Gamma_{JA}.
\end{aligned} \tag{4.10}$$

We first fix \mathcal{X}_{KB}^s and solve (4.9) for \mathcal{X}_Ξ^s by Neumann series. Notice that the solution \mathcal{X}_Ξ^s becomes analytic w.r.t. z, Ξ, s and \mathcal{X}_{KB}^s . Next, since $\Omega(\partial_{JB}\Phi_J(z_J), \partial_{JA}\Phi_J[z_J])$ is invertible, we can solve (4.10) again by Neumann series. Therefore, we obtain \mathcal{X}_{JA}^s and \mathcal{X}_Ξ^s which satisfies (4.9), (4.10). Finally, (4.8) follows from (4.6) in Lemma 4.2 and the gauge invariance of \mathcal{X} follows from the gauge invariance of Ω and Ω_0 . \square

Lemma 4.4. *For the $d_1 > 0$ and $\mathcal{X}^s(z, \Xi)$ of Lemma 4.3, consider the following system, which is well defined in $(s, z, \Xi) \in (-4, 4) \times D_{\mathbb{C}^n \times \Sigma_c^{-r_0}}(0, d_1)$:*

$$\frac{d}{ds}S_J = \mathcal{X}_J^s(z + S_z, \Xi + S_\Xi) \text{ and } \frac{d}{ds}S_\Xi = \mathcal{X}_\Xi^s(z + S_z, \Xi + S_\Xi), \quad (4.11)$$

where $S_z = (S_1, \dots, S_{2n})$. Then the following facts holds.

(i) For $d_2 \in (0, d_1)$ sufficiently small system (4.11) generates flows

$$(S_z(s, z, \Xi), S_\Xi(s, z, \Xi)) \in C^\omega((-2, 2) \times D_{\mathbb{C}^n \times \Sigma_c^{-r_0}}(0, d_2), D_{\mathbb{C}^n \times \Sigma_c^{r_0}}(0, d_1)),$$

with

$$\sum_{J=1}^{2n} |S_J(s, z, \Xi)| + \|S_\Xi(s, z, \Xi)\|_{\Sigma^{r_0}} \leq c|z|(\|\mathbf{Z}\| + \|\Xi\|_{\Sigma^{-r_0}}). \quad (4.12)$$

(ii) We have $S_J(s, e^{i\theta}z, e^{i\theta}\Xi) = e^{i\theta}S_J(s, z, \Xi)$, $S_\Xi(s, e^{i\theta}z, e^{i\theta}\Xi) = e^{i\theta}S_\Xi(s, z, \Xi)$.

Proof. It is elementary to solve the system (4.11) by contraction mapping theorem or implicit function theorem. The properties in (i), (ii) comes from (i), (ii) of Lemma 4.3. \square

From Lemmas (4.2), 4.3 and 4.4 we immediately have Proposition 4.1.

Proof of Proposition 4.1. Set $\tilde{\mathfrak{F}}^D(z, \Xi) := (S_z(1, z, \Xi), S_\Xi(1, z, \Xi))$. Then, by (4.12) and (ii) of Lemma 4.4, we have (4.2) and gauge property of $\mathcal{F}^D := \text{Id} + \tilde{\mathfrak{F}}^D$. Finally, since (S_z, S_Ξ) is the solution of (4.11) we see that $\mathfrak{F}^s = \text{Id} + (S_z(s, \cdot, \cdot), S_\Xi(s, \cdot, \cdot))$ is the solution of $\frac{d}{ds}\mathcal{X}^s(\mathfrak{F}^s)$. Therefore, by (4.3), we have $(\mathfrak{F}^D)^*\Omega = \Omega_0$. \square

Consider now the symplectic form Ω_0 in (4.1). Notice that it is of the following form

$$\Omega_0 = \sum_{J=1}^{2n} 2i\tilde{\omega}_J(1 + a_J(|z_J|^2))dz_J \wedge d\bar{z}_J + 2\langle d\Xi | \mathbb{J} d\Xi \rangle \text{ for } \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (4.13)$$

for appropriate real valued functions $a_J(|z_J|^2)$ with $|a_J(|z_J|^2)| \leq c|z_J|^2$. Indeed, by direct calculation, we have

$$\begin{aligned} \Omega(d\Phi_J[z_J]X | d\Phi_J[z_J]Y) &= \Omega(\partial_{JR}\Phi_J[z_J] | \partial_{JI}\Phi_J[z_J])(X_{JR}Y_{JI} - X_{JI}Y_{JR}) \\ &= \frac{i}{2}\Omega(\partial_{JR}\Phi_J[z_J] | \partial_{JI}\Phi_J[z_J])dz \wedge d\bar{z}(X, Y). \end{aligned}$$

Next, by Lemma A.1, we can show $\frac{1}{2}\Omega(\partial_{JR}\Phi_J[z_J] | \partial_{JI}\Phi_J[z_J])$ is gauge invariant. Therefore, we can write $\frac{1}{2}\Omega(\partial_{JR}\Phi_J[z_J] | \partial_{JI}\Phi_J[z_J]) = \tilde{a}(|z_J|^2)$. Finally, the constant term is given by

$$\begin{aligned} \frac{1}{2}\Omega(\partial_{JR}\begin{pmatrix} z_J\phi_J \\ i\tilde{\omega}_J z_J\phi_J \end{pmatrix} | \partial_{JI}\begin{pmatrix} z_J\phi_J \\ i\tilde{\omega}_J z_J\phi_J \end{pmatrix}) &= \left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_J \\ i\tilde{\omega}_J\phi_J \end{pmatrix} \middle| \begin{pmatrix} i\phi_J \\ -\tilde{\omega}_J\phi_J \end{pmatrix} \right\rangle \\ &= 2\tilde{\omega}_J. \end{aligned}$$

Therefore, we have (4.13).

Remark 4.5. A schematic explanation of why we adapt Ω_0 and we do not use the constant symplectic form

$$\Omega'_0 := 2i\tilde{\omega}_J dz_J \wedge d\bar{z}_J + 2\langle d\Xi | \mathbb{J} d\Xi \rangle,$$

is the following. First, notice that by (4.11), $S_J \sim \mathcal{X}_J^0$ and $S_\Xi \sim \mathcal{X}_\Xi^0$. Further, by (4.9) and (4.10), we have $\mathcal{X}_{JA}^0 = \mathcal{A}_{KB} \Gamma_{KB}$ and $\mathcal{X}_\Xi^0 = -2^{-1} \mathbb{J} \Gamma_\Xi$, where \mathcal{A}_{KB} represents the inverse of the matrix $\Omega(\partial_{JB} \Phi_J[z_J], \partial_{JA} \Phi_J[z_J])$. Therefore, the estimate (4.2) is a direct consequence of (4.6) and if (4.6) do not hold, we cannot expect (4.2) to hold. Now, if we take Ω'_0 instead of Ω_0 , then in Γ_{JA} there will be pure terms of z_J . So we cannot have the estimate (4.6). Finally, we remark that the estimate (4.2) is needed not only to come back to the original coordinate after scattering but it is also crucial for the cancelation lemma (Lemma 5.1 below).

We introduce an index $\ell = J, \bar{J}$ with $J = 1, \dots, 2n$. Given $F \in C^1(\mathcal{U}, \mathbb{R})$ with \mathcal{U} an open subset of $\mathbb{C}^n \times \Sigma_c^{-r}$, its Hamiltonian vector field X_F . We have summing on J

$$\begin{aligned} i_{X_F} \Omega_0 &= 2i\tilde{\omega}_J(1 + a_J(|z_J|^2))((X_F)_J d\bar{z}_J - (X_F)_{\bar{J}} dz_J) + 2\langle (X_F)_\Xi | \mathbb{J} d\Xi \rangle \\ &= \partial_J F dz_J + \partial_{\bar{J}} F d\bar{z}_J + \langle \nabla_\Xi F | d\Xi \rangle. \end{aligned} \quad (4.14)$$

Comparing the components of the two sides of (4.14) for some $\varpi_J(|z_J|^2)$ with $|\varpi_J(|z_J|^2)| \leq c|z_J|^2$ we get

$$(X_F)_J = -\frac{1}{2}i\tilde{\omega}_J^{-1}(1 + \varpi_J(|z_J|^2))\partial_{\bar{J}} F, \quad (X_F)_\Xi = \frac{1}{2}\mathbb{J}\nabla_\Xi F. \quad (4.15)$$

(4.15) imply that in this latter system of coordinates equation (1.1) takes the form

$$i\dot{z}_J = \frac{1}{2}\tilde{\omega}_J^{-1}(1 + \varpi_J(|z_J|^2))\partial_{\bar{J}} E^D, \quad \dot{\Xi} = -\mathbb{J}\nabla_\Xi E^D, \quad (4.16)$$

where $E^D := (\mathfrak{F}^D)^* E^0$ which is the pull back of E^0 by \mathfrak{F}^D .

5 Effective Hamiltonian

The pullback of the energy E by the map \mathfrak{F} in Lemma 4.4 has the following expansion.

Lemma 5.1. *For $(z, \Xi) \in D_{\mathbb{C}^{2n}}(0, d_2) \times (D_{\mathbf{H}^{\frac{1}{2}}}(0, d_2) \cap \mathcal{H}_c[0])$ we have*

$$\begin{aligned} E^D(z, \Xi) &:= (\mathfrak{F}^D)^* E^0(z, \Xi) = \sum_{J=1}^{2n} E(\Phi_J[z_J]) + \langle B\Xi | \Xi \rangle + E_P(B^{-\frac{1}{2}}\eta) \\ &+ \sum_{l=1}^{\infty} \sum_{J=0}^{2n} \sum_{|m|=l+1} \mathbf{Z}^m a_{Jm}^{(1)}(|z_J|^2) + \sum_{l=1}^{\infty} \sum_{J=1}^{2n} \sum_{|m|=l} \langle z_J \mathbf{Z}^m G_{Jm}^{(1)}(|z_J|^2) | \Xi \rangle \\ &+ \sum_{i+j=2} \langle G_{2ij}^{(1)}(z, \Xi) | (B^{-\frac{1}{2}}\eta)^i \overline{B^{-\frac{1}{2}}\eta}^j \rangle + \sum_{i+j=3} \langle G_{3ij}^{(1)}(z, \Xi) | B^{-\frac{1}{2}}\eta^i \overline{B^{-\frac{1}{2}}\eta}^j \rangle + \mathcal{R}^{(1)}, \end{aligned} \quad (5.1)$$

where $a_{Jm}^{(1)}, G_{Jm}^{(1)}, G_{2ij}^{(1)}, G_{3ij}^{(1)}$ and $\mathcal{R}^{(1)}$ satisfies the condition (1)–(3) in Lemma 3.1.

Remark 5.2. The only difference between (3.3) and (5.1) are that in the second line the terms with $l = 0$ vanishes in (5.1).

Proof. First, we can write $E^D(z, \Xi) = E(\tilde{\Psi}[z, \Xi] + B^{-\frac{\sigma_3}{2}} \Xi)$ for $\tilde{\Psi}[z, \Xi] = (\mathfrak{F}^D)^* \mathcal{F}(z, \Xi) - B^{-\frac{\sigma_3}{2}} \Xi$. Notice that $\tilde{\Psi} \in C^\omega(D_{\mathbb{C}^{2n} \times \Sigma^{-r_0}}(0, d_2), \Sigma^{r_0})$. Therefore, the proof of the expansion becomes completely parallel to the proof of (3.1).

The only nontrivial thing remaining is the absence of the terms with $l = 0$ in (5.1). To show this fix J and notice that by (4.2) a solution of (1.1) with initial value (z, Ξ) (in the Darboux coordinate) with $\Xi = 0$ and $z_K = 0$ (for $K \neq J$) is the nonlinear bound state given in Proposition 1.1. Therefore, if we have $\Xi(0) = 0$ and $z_K(0) = 0$ ($K \neq J$), we have $\Xi(t) = 0$ and $z_K(t) = 0$ ($K \neq J$) for all times. In particular, we have $\frac{d}{dt}\big|_{t=0} \Xi(t) = 0$ and $\frac{d}{dt}\big|_{t=0} z_K(t) = 0$.

We first show $z_J G_{J0}^{(1)}(|z_J|^2) = 0$ for all z_J . Suppose $z_J G_{J0}^{(1)}(|z_J|^2) \neq 0$ for some z_J . Then, taking the initial data such as $\Xi(0) = 0$, $z_K(0) = 0$ ($K \neq J$) and $z_J(0) = z_J$, we have

$$\frac{d}{dt}\bigg|_{t=0} \Xi(t) = -iz_J G_{J0}(|z_J|^2) \neq 0,$$

contradicting $\Xi(t) = 0$ for all times.

Next, we show $a_{J\mathbf{m}}^{(1)}(|z_J|^2) = 0$ for all $|\mathbf{m}| = 1$. Notice that $a_{0\mathbf{m}}^{(1)} = 0$ since E^0 has no such term by the orthogonality of ϕ_J and ϕ_K ($J \neq K$). Furthermore, setting $\mathbf{Z}^{\mathbf{m}} = z^L \bar{z}^K$, we can assume that $L = J$ or $K = J$. Indeed, suppose $L \neq J$ and $K = J$. Notice that we have just shown $a_{J\mathbf{m}}^{(1)}(|z_J|^2)|_{z_J=0} = 0$. Then we can write $\mathbf{Z}^{\mathbf{m}} a_{J\mathbf{m}}^{(1)}(|z_J|^2) = (z_L \bar{z}_J)(z_J \bar{z}_K)(|z_J|^{-2} a_{J\mathbf{m}}(|z_J|^2))$ and move this among the terms with $|\mathbf{m}| = 2$.

So, without loss of generality, we can assume $L = J$. Now, suppose that $a_{J\mathbf{m}}(|z_J|^2) \neq 0$. Then, taking the initial data such as $\Xi(0) = 0$, $z_K(0) = 0$ ($K \neq J$) and $z_J(0) = z_J$, we have

$$2i \frac{d}{dt}\bigg|_{t=0} z_K(t) = \tilde{\omega}_K^{-1} z_J a_{J\mathbf{m}}^{(1)}(|z_J|^2) \neq 0$$

contradicting $z_K(t) = 0$ for all times. \square

To extract an effective Hamiltonian we cancel as many terms as possible from (5.1) by means of a normal forms argument. The following result is proved in [9].

Proposition 5.3 (Birkhoff normal forms). *Assume (H1)–(H3). There exists $r_B = r_B(r_0)$ and $d_B > 0$ with $r_B(r_0) \xrightarrow{r_0 \rightarrow \infty} \infty$ such that there exists $\tilde{\mathfrak{F}}^B = (\tilde{\mathfrak{F}}_z^B, \tilde{\mathfrak{F}}_\Xi^B) \in C^\omega(D_{\mathbb{C}^{2n} \times \Sigma^{-r_B}}(0, d_B), \mathbb{C}^{2n} \times \Sigma^{r_B})$ such that*

$$|\tilde{\mathfrak{F}}_z^B(z, \Xi)| + \|\tilde{\mathfrak{F}}_\Xi^B(z, \Xi)\|_{\Sigma^{r_B}} \leq c|z|(|\mathbf{Z}| + \|\Xi\|_{\Sigma^{-r_B}}), \quad (5.2)$$

and $\mathfrak{F}^B := \text{Id} + \tilde{\mathfrak{F}}^B$ satisfies $(\mathfrak{F}^B)^* \Omega_0 = \Omega_0$. In addition, we have $\mathfrak{F}^B(e^{i\theta} z, e^{i\theta} \Xi) = e^{i\theta} \mathfrak{F}^B(z, \Xi)$. Further, setting $E^B := (\mathfrak{F}^B)^* E^D$, we have

$$\begin{aligned} E^B(z, \Xi) &= \sum_{J=1}^{2n} E(\Phi_J[z_J]) + \langle B\Xi | \Xi \rangle + E_P(B^{-\frac{1}{2}} \eta) \\ &+ \sum_{J=0}^{2n} \sum_{\mathbf{m} \in \mathcal{M}_0(2N+4)} \mathbf{Z}^{\mathbf{m}} a_{J\mathbf{m}}(|z_J|^2) + \sum_{J=1}^{2n} \sum_{\mathbf{m} \in \mathcal{M}(2N+3)} \langle z_J \mathbf{Z}^{\mathbf{m}} G_{J\mathbf{m}}(|z_J|^2) | \Xi \rangle + \mathcal{R}, \end{aligned} \quad (5.3)$$

where

$$\begin{aligned} \mathcal{R} = & \sum_{l=2N+4}^{\infty} \sum_{J=0}^{2n} \sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} a_{J\mathbf{m}}(|z_J|^2) + \sum_{l=2N+4}^{\infty} \sum_{J=1}^{2n} \sum_{|\mathbf{m}|=l} \langle z_J \mathbf{Z}^{\mathbf{m}} G_{J\mathbf{m}}(|z_J|^2) | \Xi \rangle \\ & + \sum_{i+j=2} \langle G_{2ij}(z, \Xi) | (B^{-\frac{1}{2}} \eta)^i \overline{B^{-\frac{1}{2}} \eta}^j \rangle + \sum_{i+j=3} \langle G_{3ij}(z, \Xi) | B^{-\frac{1}{2}} \eta^i \overline{B^{-\frac{1}{2}} \eta}^j \rangle + \mathcal{R}^B, \end{aligned} \quad (5.4)$$

with $a_{J\mathbf{m}}$, $G_{J\mathbf{m}}$, G_{2ij} , G_{3ij} and \mathcal{R}^B satisfies the condition (1)–(3) in Lemma 3.1.

□

We now make a further change of coordinates introducing $\Xi = A\Theta$ where

$$\mathbb{J} = -A i \sigma_3 A^{-1} \text{ for } A = 2^{-\frac{1}{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}. \quad (5.5)$$

We have $A^* = A^{-1}$. So now we use (z, Θ) as coordinates. In this new coordinate system the Hamiltonian (5.3) takes the form

$$\begin{aligned} \mathcal{H}(z, \Theta) &= \sum_{J=1}^{2n} E(\Phi_J[z_J]) + \langle B\Theta | \Theta \rangle + Z(z, \mathbf{Z}, \Theta) + \mathcal{R}, \text{ where } Z = \mathcal{Z}_0 + \mathcal{Z}_1, \\ \mathcal{Z}_0 &= \sum_{\mathbf{m} \in \mathcal{M}_0(2N+4)} \sum_{J=0}^{2n} \mathbf{Z}^{\mathbf{m}} a_{J\mathbf{m}}(|z_J|^2), \\ \mathcal{Z}_1 &= \sum_{J=1}^{2n} \sum_{\mathbf{m} \in \mathcal{M}_J(2N+3)} \langle z_J \mathbf{Z}^{\mathbf{m}} G_{J\mathbf{m}}(|z_J|^2) | \Theta \rangle = \sum_{(\mu, \nu) \in M} \langle z^\mu \bar{z}^\nu G_{\mu\nu}(|z_J|^2) | \Theta \rangle, \end{aligned} \quad (5.6)$$

where $\tilde{J} = \tilde{J}(\mu, \nu)$ is determined from the correspondence between $\mathcal{M}_J(2N+3)$ and M . The symplectic form Ω_0 takes the form

$$2i\tilde{\omega}_J(1 + a_J(|z_J|^2))dz_J \wedge d\bar{z}_J + 2\text{Im}\langle i\sigma_3 d\Theta | d\Theta \rangle, \quad (5.7)$$

and system (4.16) becomes, for $1 + \varpi_J(|z_J|^2) := (1 + a_J(|z_J|^2))^{-1}$,

$$i\dot{z}_J = \frac{1}{2}\tilde{\omega}_J^{-1}(1 + \varpi_J(|z_J|^2))\partial_{\bar{z}_J}\mathcal{H} \quad , \quad i\dot{\Theta} = \frac{1}{2}\sigma_3 \nabla_{\Theta}\mathcal{H}. \quad (5.8)$$

The final step needed to prove Theorem 1.5 is the following.

Proposition 5.4 (Main Estimates). *There exist $\epsilon_0 > 0$ and $C_0 > 0$ s.t. if the constant $0 < \epsilon$ of Theorem 1.5 satisfies $\epsilon < \epsilon_0$, for $I = [0, \infty)$ and $C = C_0$ we have:*

$$\|\Theta\|_{L_t^p([0, \infty), W_x^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}, q} \times W_x^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}, q})} \leq C\epsilon \text{ for all admissible pairs } (p, q), \quad (5.9)$$

$$\|z^\mu \bar{z}^\nu\|_{L_t^2([0, \infty))} \leq C\epsilon \text{ for all } (\mu, \nu) \in M_{\min}, \quad (5.10)$$

$$\|z_J\|_{W_t^{1, \infty}[0, \infty)} \leq C\epsilon \text{ for all } J \in \{1, \dots, 2n\}. \quad (5.11)$$

Furthermore, there exists $\rho_+ \in [0, \infty)^{2n}$ s.t. there exist a j_0 with $\rho_{+j} = 0$ for $j \neq j_0$, and there exists $\eta_+ \in H^1$ s.t. $|\rho_+ - |z(0)|| \leq C\epsilon$ and $\Theta_+ \in H^{\frac{1}{2}} \times H^{\frac{1}{2}}$ with $\|\Theta_+\|_{H^{\frac{1}{2}} \times H^{\frac{1}{2}}} \leq C\epsilon$, such that

$$\lim_{t \rightarrow +\infty} \|\Theta(t) - e^{i\sigma_3 t \sqrt{-\Delta + m^2}} \Theta_+\|_{H^{\frac{1}{2}} \times H^{\frac{1}{2}}} = 0 \quad , \quad \lim_{t \rightarrow +\infty} |z_j(t)| = \rho_{+j}. \quad (5.12)$$

Proof of Theor.1.5. Before proving Prop.5.4 we will show that it implies Theor.1.5. The proof is similar to the analogous step in [9]. Denote by (z', Ξ') the initial coordinate system. By (4.2) and (5.2), we have

$$|z' - z| + \|\Xi' - A\Theta\|_{\Sigma^r} \leq c|z| (|\mathbf{Z}| + \|\Theta\|_{\Sigma^{-r}}). \quad (5.13)$$

(5.12) implies $\lim_{t \rightarrow +\infty} \mathbf{Z}(t) = 0$ and by elementary arguments for $s > 3/2$ we have

$$\lim_{t \rightarrow +\infty} \|e^{i\sigma_3 t \sqrt{-\Delta + m^2}} \Theta_+\|_{L^2, -s}(\mathbb{R}^3) = 0 \text{ for any } \Xi_+ \in L^2. \quad (5.14)$$

These two limits, and (5.12) and (5.13) imply

$$\lim_{t \rightarrow +\infty} |z'(t) - z(t)| + \|\Xi'(t) - A\Theta(t)\|_{\Sigma^r} = 0.$$

Then we have the limit

$$\lim_{t \rightarrow +\infty} |z'_j(t)| = \lim_{t \rightarrow +\infty} |z_j(t)| = \rho_{+j}. \quad (5.15)$$

Next, in $H^1 \times L^2$ by (5.5) we have

$$0 = \lim_{t \rightarrow +\infty} B^{-\frac{\sigma_3}{2}} [\Xi'(t) - A e^{i\sigma_3 t \sqrt{-\Delta + m^2}} \Theta_+] = \lim_{t \rightarrow +\infty} B^{-\frac{\sigma_3}{2}} [\Xi'(t) - e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} \Xi_+].$$

Setting $\Xi_+ := A\Theta_+$ write

$$B^{-\frac{\sigma_3}{2}} e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} \Xi_+ = e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} e^{\mathbb{J}t \sqrt{-\Delta + m^2}} e^{-\mathbb{J}t B} B^{-\frac{\sigma_3}{2}} e^{\mathbb{J}t B} e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} \Xi_+$$

and consider the wave operators

$$W = s - \lim_{t \rightarrow +\infty} e^{\mathbb{J}t B} e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} \text{ and } Z = s - \lim_{t \rightarrow +\infty} e^{\mathbb{J}t \sqrt{-\Delta + m^2}} e^{-\mathbb{J}t B}.$$

Then we have

$$\lim_{t \rightarrow +\infty} [B^{-\frac{\sigma_3}{2}} e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} \Xi_+ - e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} Z B^{-\frac{\sigma_3}{2}} W \Xi_+] = 0.$$

Since $Z B^{-\frac{\sigma_3}{2}} W \Xi_+ = (-\Delta + m^2)^{-\frac{\sigma_3}{4}} \Xi_+$ we conclude that in $H^{\frac{1}{2}} \times H^{\frac{1}{2}}$

$$\lim_{t \rightarrow \infty} [B^{-\frac{\sigma_3}{2}} \Xi'(t) - e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} (-\Delta + m^2)^{-\frac{\sigma_3}{4}} \Xi_+] = 0.$$

On the other hand for

$$u_+ = (-\Delta + m^2)^{-\frac{1}{4}} (\Xi_+)_1 \in H^1 \text{ and } v_+ = (-\Delta + m^2)^{\frac{1}{4}} (\Xi_+)_2 v_+ \in L^2,$$

where $\Xi_+ = ((\Xi_+)_1, (\Xi_+)_2)$, we have

$$e^{-\mathbb{J}t \sqrt{-\Delta + m^2}} (-\Delta + m^2)^{-\frac{\sigma_3}{4}} \Xi_+ = \cos(t \sqrt{-\Delta + m^2}) \begin{pmatrix} u_+ \\ v_+ \end{pmatrix} - \begin{pmatrix} -\frac{\sin(t \sqrt{-\Delta + m^2})}{\sqrt{-\Delta + m^2}} v_+ \\ \sqrt{-\Delta + m^2} \sin(t \sqrt{-\Delta + m^2}) u_+ \end{pmatrix}$$

which yields the 1st line in (1.13).

We have formula (1.12) with z replaced by z' and

$$\begin{aligned} (\eta, \xi) &= R[z'] \Xi' = B^{-\frac{\sigma_3}{2}} A\Theta + \mathbf{A} \text{ where} \\ \mathbf{A} &= (R[z'] - B^{-\frac{\sigma_3}{2}}) A\Theta + R[z'] (\Xi' - A\Theta). \end{aligned}$$

Then by (5.12) and (5.15) we conclude the \mathbf{A} satisfies (1.15).
Finally we prove the 2nd line (1.14). We have

$$\dot{z}'_J + i\omega_J z'_J = \dot{z}_J + i\omega_J z_J + \frac{d}{dt}\tilde{\mathfrak{F}}_z(z, \Theta) + i\omega_J \tilde{\mathfrak{F}}_z(z, \Theta),$$

where $\tilde{\mathfrak{F}} := \mathfrak{F}^B \circ \mathfrak{F}^D - \text{Id}$. We need to prove that this is $O(\epsilon^2)$. We have $\dot{z}_J + i\omega_J z_J = O(\epsilon^2)$ by (6.10) below and we have $\tilde{\mathfrak{F}}_z(z, \Theta) = O(\epsilon^2)$ by Propositions 4.1 and 5.3. For the third term, we have

$$\frac{d}{dt}\tilde{\mathfrak{F}}_z(z, \Theta) = \sum_{J=1}^{2n} \sum_{A=R,I} \partial_{JA} \tilde{\mathfrak{F}}_z(z, \Theta) \dot{z}_{JA} + d_{\Theta} \tilde{\mathfrak{F}}_z(z, \Theta) \cdot \dot{\Theta},$$

with $d_{\Theta} \tilde{\mathfrak{F}}_z(z, \Theta)$ the partial derivative in Θ . Then, by

$$|\partial_{JA} \tilde{\mathfrak{F}}_z(z, \Theta)| + \|d_{\Theta} \tilde{\mathfrak{F}}_z(z, \Theta)\|_{\Sigma^r} \leq C(|z| + \|\Theta\|_{\Sigma^{-r}}),$$

and equations (6.4) and (6.10) below, we have $\frac{d}{dt}\tilde{\mathfrak{F}}_z(z, \Theta) = O(\epsilon^2)$. This yields the inequality claimed in the second line in (1.14). \square

By a standard argument (5.9)–(5.11) for $I = [0, \infty)$ are a consequence of the following Proposition.

Proposition 5.5. *There exists a constant $c_0 > 0$ such that for any $C_0 > c_0$ there is a value $\epsilon_0 = \epsilon_0(C_0)$ such that if for some $T > 0$ we have*

$$\|\Theta\|_{L_t^p([0,T], W_x^{1/q-1/p, q})} \leq C_0 \epsilon \text{ for all admissible pairs } (p, q) \quad (5.16)$$

$$\|z^\mu \bar{z}^\nu\|_{L_t^2([0,T])} \leq C_0 \epsilon \text{ for all } (\mu, \nu) \in M_{\min}, \quad (5.17)$$

then (5.16)–(5.17) hold for $C = C_0/2$.

6 Proof of Proposition 5.5

The first step in the proof is the following.

Proposition 6.1. *Assume (5.16)–(5.17). Then there exist constants c and $\epsilon_0 > 0$ s.t. for $\epsilon \in (0, \epsilon_0)$ then we have*

$$\|\Theta\|_{L_t^p([0,T], W_x^{1/q-1/p, q})} \leq c\epsilon + c \sum_{(\mu, \nu) \in M_{\min}} |z^\mu \bar{z}^\nu|_{L_t^2([0,T])} \text{ for all admissible pairs } (p, q). \quad (6.1)$$

6.1 Proof of Proposition 6.1

The facts collected in the following lemma are proved in [2].

Lemma 6.2. *Assume (H1)–(H2).*

(1) *There is a fixed constant c_0 such that for any two admissible pairs (p, q) and (a, b) we have*

$$\begin{aligned} \|e^{\pm itB} P_c u_0\|_{L^p(\mathbb{R}, W^{\frac{1}{q}-\frac{1}{p}, q}(\mathbb{R}^3))} &\leq c_0 \|u_0\|_{H^{1/2}(\mathbb{R}^3)} \\ \left\| \int_{s < t} e^{\pm i(s-t)B} P_c F(s) ds \right\|_{L^p(\mathbb{R}, W^{\frac{1}{q}-\frac{1}{p}, q}(\mathbb{R}^3))} &\leq c_0 \|F\|_{L^{a'}(\mathbb{R}, W^{\frac{1}{a}-\frac{1}{b}+1, b'}(\mathbb{R}^3))}. \end{aligned} \quad (6.2)$$

- (2) For $u_0 \in H^{2,s}(\mathbb{R}^3, \mathbb{C})$ for $s > 1/2$ and $\lambda > m$ then $R_B^\pm(\lambda)u_0 := \lim_{\varepsilon \rightarrow 0^\pm} R_B(\lambda + i\varepsilon)u_0$ is well defined in $L^{2,-s}(\mathbb{R}^3, \mathbb{C})$.
- (3) For any $s > 1$ there is a fixed $c_0 = c_0(s, a)$ such that for any admissible pair (p, q) we have

$$\left\| \int_0^t e^{\pm i(t'-t)B} P_c F(t') dt' \right\|_{L^p(\mathbb{R}, W^{\frac{1}{q}-\frac{1}{p}, q}(\mathbb{R}^3))} \leq c_0 \|B^{\frac{1}{2}} P_c F\|_{L^a(\mathbb{R}, L^{2,s}(\mathbb{R}^3))} \quad (6.3)$$

where for $p > 2$ we can pick any $a \in [1, 2]$ while for $p = 2$ we pick $a \in [1, 2)$.

□

Now we look at the equation for Θ . Then for $G_{\mu\nu} = G_{\mu\nu}(0)$ we have

$$\begin{aligned} i\dot{\Theta} &= \frac{1}{2}\sigma_3 \nabla_\Theta \mathcal{H} = \sigma_3 B \Theta + \frac{1}{2}\sigma_3 \sum_{(\mu, \nu) \in M_{min}} z^\mu \bar{z}^\nu G_{\mu\nu} + \mathbb{A} \text{ where} \\ \mathbb{A} &:= \frac{1}{2}\sigma_3 \left(\sum_{(\mu, \nu) \in M_{min}} z^\mu \bar{z}^\nu [G_{\mu\nu}(|z_j|^2) - G_{\mu\nu}] + \sum_{(\mu, \nu) \in M \setminus M_{min}} z^\mu \bar{z}^\nu G_{\mu\nu} + \nabla_\Theta \mathcal{R} \right). \end{aligned} \quad (6.4)$$

The following lemma is proved in Lemma 7.5 [2].

Lemma 6.3. Assume (5.16)–(5.17), and fix a large $s > 0$. Then there is a constant $C = C(C_0)$ independent of ϵ such that the following is true: we have $\mathbb{A} = R_1 + R_2$ with

$$\|R_1\|_{L^1([0, T], H^{\frac{1}{2}}(\mathbb{R}^3))} + \|B^{\frac{1}{2}} P_c R_2\|_{L^2 \frac{N+1}{N+2}([0, T], L^{2,s}(\mathbb{R}^3))} \leq C(C_0)\epsilon^2. \quad (6.5)$$

□

Proof of Proposition 6.1. Using Lemma 6.3 we write

$$\Theta = e^{-i\sigma_3 B t} \Theta(0) - \frac{1}{2} i \sigma_3 \sum_{(\mu, \nu) \in M_{min}} \int_0^t z^\mu \bar{z}^\nu e^{-i\sigma_3 B(s-t)} G_{\mu\nu} ds - i \sum_{j=1}^2 \int_0^t e^{i\sigma_3 B(s-t)} P_c R_j ds.$$

By (6.2) for $(a, b) = (\infty, 2)$ and (6.5)

$$\left\| \int_0^t e^{i\sigma_3 B(s-t)} R_1 ds \right\|_{L^p([0, T], W^{\frac{1}{q}-\frac{1}{p}, q}(\mathbb{R}^3))} \leq C \|R_1\|_{L_t^1([0, T], H^{\frac{1}{2}}(\mathbb{R}^3))} \leq C(C_0)\epsilon^2.$$

By (6.3) and (6.5), we get for $s > 1$

$$\left\| \int_0^t e^{i\sigma_3 B(s-t)} P_c R_2 ds \right\|_{L_t^p([0, T], W^{\frac{1}{q}-\frac{1}{p}, q}(\mathbb{R}^3))} \leq C \|\sqrt{B} P_c R_2\|_{L_t^2 \frac{N+1}{N+2}([0, T], L^{2,s}(\mathbb{R}^3))} \leq C(C_0)\epsilon^2.$$

Finally, we have

$$\begin{aligned} \left\| \int_0^t z^\mu \bar{z}^\nu e^{-i\sigma_3 B(s-t)} G_{\mu\nu} \right\|_{L^p([0, T], W^{\frac{1}{q}-\frac{1}{p}, q}(\mathbb{R}^3))} &\leq C_0 \|z^\mu \bar{z}^\nu G_{\mu\nu}\|_{L^2([0, T], W^{\frac{1}{3}+\frac{1}{2}, \frac{6}{5}}(\mathbb{R}^3))} \\ &\leq c_0 \|z^\mu \bar{z}^\nu\|_{L^2[0, T]}. \end{aligned}$$

□

Now, following a standard argument, we introduce a new variable g setting

$$g = \Theta + Y \text{ with } Y := -\frac{1}{2} \sum_{(\alpha, \beta) \in M_{min}} z^\alpha \bar{z}^\beta R_{-\sigma_3 B}^+(\tilde{\omega} \cdot (\beta - \alpha)) \sigma_3 G_{\alpha\beta}. \quad (6.6)$$

Substituting (6.6) in (6.4) we obtain

$$i\dot{g} = \sigma_3 Bg + i\dot{Y} - \sigma_3 BY + \frac{1}{2} \sigma_3 \sum_{(\mu, \nu) \in M_{min}} z^\mu \bar{z}^\nu G_{\mu\nu} + \mathbb{A}, \quad (6.7)$$

where $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_{2n})$ has been defined in (1.4). We then compute

$$i\dot{Y} = \sigma_3 BY - \frac{1}{2} \sigma_3 \sum_{(\mu, \nu) \in M_{min}} z^\mu \bar{z}^\nu G_{\mu\nu} + \mathbf{T}, \text{ where}$$

$$\mathbf{T} := \sum_{J=1}^{2n} [\partial_{z_J} Y (i\dot{z}_J - \tilde{\omega}_J z_J) + \partial_{\bar{z}_J} Y (i\dot{\bar{z}}_J + \tilde{\omega}_J \bar{z}_J)].$$

Then

$$i\dot{g} = \sigma_3 Bg + \mathbf{T} + \mathbb{A} \quad (6.8)$$

where \mathbf{T} is smaller than the quantities which have been canceled out. The following result is proved in [2].

Lemma 6.4. *Assume (5.16)–(5.17). Fix $s > 9/2$. Then, there are constants $\epsilon_0 > 0$ and $C > 0$ such that, for $\epsilon \in (0, \epsilon_0)$ and for c_0 the constant in Lemma 6.2, we have*

$$\|g\|_{L_t^2([0, T], H^{-4, -s}(\mathbb{R}^3))} \leq c_0 \epsilon + C \epsilon^2. \quad (6.9)$$

□

6.2 Estimate of the discrete variables z

In the following, we use $(g|h) = \int_{\mathbb{R}^3} (g_1(x) \overline{h_1(x)} + g_2(x) \overline{h_2(x)}) dx$.

Let us turn now to the analysis of the Fermi Golden Rule (FGR). The equation of z is

$$i\dot{z}_J = (2\tilde{\omega}_J)^{-1} (1 + \varpi_J(|z_J|^2)) \left\{ (\partial_{\bar{z}_J} E(\Phi_J[z_J] + \partial_{\bar{z}_J} \mathcal{Z}_0(z))) \right. \\ \left. + \frac{1}{2} \sum_{(\mu, \nu) \in M_{min}} \left[\frac{z^\mu \bar{z}^\nu}{\bar{z}_J} \nu_J (G_{\mu\nu}|\Theta) + \frac{\bar{z}^\mu z^\nu}{\bar{z}_J} \mu_J (\overline{G_{\mu\nu}}|\overline{\Theta}) \right] + \mathfrak{E}_J \right\}, \quad (6.10)$$

and

$$\mathfrak{E}_J := \frac{1}{2} \sum_{(\mu, \nu) \in M \setminus M_{min}} \left[\frac{z^\mu \bar{z}^\nu}{\bar{z}_J} \nu_J (G_{\mu\nu}(|z_{\bar{J}}|^2)|\Theta) + \frac{\bar{z}^\mu z^\nu}{\bar{z}_J} \mu_J (\overline{G_{\mu\nu}}(|z_{\bar{J}}|^2)|\overline{\Theta}) \right] + \partial_{\bar{z}_J} \mathcal{R} \\ + \frac{1}{2} \sum_{(\mu, \nu) \in M_{min}} \nu_J \frac{z^\mu \bar{z}^\nu}{\bar{z}_J} (G_{\mu\nu}(|z_{\bar{J}}|^2) - G_{\mu\nu}|\Theta) + \frac{1}{2} \sum_{(\mu, \nu) \in M_{min}} \mu_J \frac{z^\nu \bar{z}^\mu}{\bar{z}_J} (\overline{G_{\mu\nu}}(|z_{\bar{J}}|^2) - \overline{G_{\mu\nu}}|\overline{\Theta}) \\ + \sum_{(\mu, \nu) \in M} \delta_{J\bar{J}} z_J \langle z^\mu \bar{z}^\nu G'_{\mu\nu}(|z_{\bar{J}}|^2)|\Theta \rangle.$$

Estimates (5.16)–(5.17) imply the following simple lemma, see [9] for the proof.

Lemma 6.5. *For $\epsilon_0 > 0$ sufficiently small there is a fixed c_1 s.t. for $\epsilon \in (0, \epsilon_0)$ we have*

$$\|\mathfrak{E}_J z_J\|_{L^1[0,T]} \leq c_1 \epsilon^2. \quad (6.11)$$

□

Now, we substitute $\Theta = g - Y$ from (6.6) obtaining

$$\begin{aligned} i\dot{z}_J = & \frac{1 + \varpi_J(|z_J|^2)}{2\tilde{\omega}_J} \left\{ \partial_{\bar{z}_J} E(\Phi_J[z_J] + \partial_{\bar{z}_J} Z_0(z)) \right. \\ & + \frac{1}{4} \sum_{\substack{(\mu, \nu) \in M_{min} \\ (\alpha, \beta) \in M_{min}}} \left[\nu_J \frac{z^{\mu+\beta} \bar{z}^{\nu+\alpha}}{\bar{z}_J} (G_{\mu\nu} |R_{-\sigma_3 B}^+(\tilde{\omega} \cdot (\beta - \alpha)) \sigma_3 G_{\alpha\beta}) \right. \\ & \left. \left. + \mu_J \frac{z^{\nu+\alpha} \bar{z}^{\mu+\beta}}{\bar{z}_J} (\bar{G}_{\mu\nu} |R_{-\sigma_3 B}^-(\tilde{\omega} \cdot (\beta - \alpha)) \sigma_3 \bar{G}_{\alpha\beta}) \right] + \mathcal{F}_J \right\}, \text{ where} \end{aligned} \quad (6.12)$$

$$\mathcal{F}_J := \frac{1 + \varpi_J(|z_J|^2)}{4\tilde{\omega}_J} \left\{ \sum_{(\mu, \nu) \in M_{min}} \left[\frac{z^\mu \bar{z}^\nu}{\bar{z}_J} \nu_J (G_{\mu\nu} |g) + \frac{\bar{z}^\mu z^\nu}{\bar{z}_J} \mu_J (\bar{G}_{\mu\nu} |\bar{g}) \right] + \mathfrak{E}_J \right\}.$$

From (6.9) and (6.11) we get the following simple bound.

Lemma 6.6. *For $\epsilon_0 > 0$ sufficiently small there is a fixed c_1 s.t. for $\epsilon \in (0, \epsilon_0)$ we have*

$$\|\mathcal{F}_J z_J\|_{L^1[0,T]} \leq (1 + C_0) c_1 \epsilon^2. \quad (6.13)$$

□

We now introduce a new variable ζ defined by $\zeta_J = z_J + T_J(z)$ where

$$\begin{aligned} T_J(z) := & -\frac{1}{8} \sum_{\substack{(\mu, \nu) \in M_{min} \\ (\alpha, \beta) \in M_{min}}} \frac{\nu_J z^{\mu+\beta} \bar{z}^{\nu+\alpha}}{((\mu - \nu) \cdot \tilde{\omega} - (\alpha - \beta) \cdot \tilde{\omega}) \bar{z}_J} (G_{\mu\nu} |R_{-\sigma_3 B}^+(\tilde{\omega} \cdot (\beta - \alpha)) \sigma_3 G_{\alpha\beta}) \\ & - \frac{1}{8} \sum_{\substack{(\mu, \nu) \in M_{min} \\ (\alpha, \beta) \in M_{min}}} \frac{\mu_J z^{\nu+\alpha} \bar{z}^{\mu+\beta}}{((\alpha - \beta) \cdot \tilde{\omega} - (\mu - \nu) \cdot \tilde{\omega}) \bar{z}_J} (\bar{G}_{\mu\nu} |R_{-\sigma_3 B}^-(\tilde{\omega} \cdot (\beta - \alpha)) \sigma_3 \bar{G}_{\alpha\beta}), \end{aligned} \quad (6.14)$$

with the summation performed over the pairs where the formula makes sense, that is $(\mu - \nu) \cdot \tilde{\omega} \neq (\alpha - \beta) \cdot \tilde{\omega}$. It is easy to see that

$$\|\zeta - z\|_{L^2(0,T)} \leq c(N, C_0) \epsilon^2 \text{ and } \|\zeta - z\|_{L^\infty(0,T)} \leq c(N, C_0) \epsilon^2. \quad (6.15)$$

We set $\Lambda := \{(\nu - \mu) \cdot \tilde{\omega} : (\mu, \nu) \in M_{min}\}$. For any $L \in \Lambda$ set

$$M_L := \{(\mu, \nu) \in M_{min} : (\nu - \mu) \cdot \tilde{\omega} = L\}. \quad (6.16)$$

Then all terms of (6.12) where $(\mu - \nu) \cdot \tilde{\omega} \neq (\alpha - \beta) \cdot \tilde{\omega}$ cancel out giving an equation like (6.17) below, which is the equation satisfied by ζ and involves an error term which by [9] satisfies (6.18).

Lemma 6.7. ζ satisfies equations

$$\begin{aligned} i\dot{\zeta}_J = \frac{1 + \varpi_J(|\zeta_J|^2)}{2\tilde{\omega}_J} & \left\{ \partial_{\bar{\zeta}_J} E(\Phi_J[\zeta_J]) + \partial_{\bar{\zeta}_J} \mathcal{Z}_0(\zeta) + \right. \\ \frac{1}{4} \sum_{L \in \Lambda} \sum_{\substack{(\mu, \nu) \in M_L \\ (\alpha, \beta) \in M_L}} & \left(\nu_J \frac{\zeta^{\mu+\beta} \bar{\zeta}^{\nu+\alpha}}{\bar{\zeta}_J} (G_{\mu\nu} |R_{-\sigma_3 B}^+(L) \sigma_3 G_{\alpha\beta}) + \mu_J \frac{\zeta^{\nu+\alpha} \bar{\zeta}^{\mu+\beta}}{\bar{\zeta}_J} (\bar{G}_{\mu\nu} |R_{-\sigma_3 B}^-(L) \sigma_3 \bar{G}_{\alpha\beta}) \right) \\ & \left. + \mathcal{G}_J \right\}, \end{aligned} \quad (6.17)$$

where there are fixed c_1 and $\epsilon_0 > 0$ such that for $T > 0$ and $\epsilon \in (0, \epsilon_0)$ we have

$$\sum_{J=1}^{2n} |\tilde{\omega}| \|\mathcal{G}_J \zeta_J\|_{L^1[0, T]} \leq (1 + C_0) c_1 \epsilon^2. \quad (6.18)$$

We multiply (6.17) by $\bar{\zeta}_J \tilde{\omega}_J^2 (1 + a_J(|\zeta_J|^2))$, see (5.7). Let

$$A_J(s) := 4 \int_0^s \tilde{\omega}_J^2 (1 + a_J(s)) ds.$$

Near 0, since $a_J(s) = O(s)$, we have $A_J(s) = \tilde{\omega}_J^2 s + O(s^2)$. By (6.17), we obtain

$$\begin{aligned} \frac{1}{4} \sum_{J=1}^{2n} \frac{d}{dt} A_J(|\zeta_J|^2) &= \sum_{J=1}^{2n} \text{Im}[\tilde{\omega}_J \partial_{\bar{\zeta}_J} E(\Phi_J[\zeta_J]) \bar{\zeta}_J] + \sum_{J=1}^{2n} \text{Im}[\tilde{\omega}_J \partial_{\bar{\zeta}_J} \mathcal{Z}_0(\zeta)] \\ &+ \sum_{J=1}^{2n} \left(\frac{\tilde{\omega}_J}{4} \sum_{L \in \Lambda} \sum_{\substack{(\mu, \nu) \in M_L \\ (\alpha, \beta) \in M_L}} \text{Im} \left((\nu_J - \mu_J) \zeta^{\mu+\beta} \bar{\zeta}^{\nu+\alpha} (G_{\mu\nu} |R_{-\sigma_3 B}^+(L) \sigma_3 G_{\alpha\beta}) \right) + \tilde{\omega}_J \text{Im}(\mathcal{G}_J \bar{\zeta}_J) \right). \end{aligned}$$

The 1st in the r.h.s. equals 0 since $E(\Phi_J[\zeta_J]) = F_J(|\zeta_J|^2)$ for F_J real valued. Also the 2nd term in the r.h.s. equals 0. Indeed,

$$\begin{aligned} \sum_J \tilde{\omega}_J (\bar{z}_J \partial_{\bar{z}_J} \mathcal{Z}_0 - z_J \partial_{z_J} \mathcal{Z}_0) &= \\ \sum_{\mathbf{m}, K} a_{K\mathbf{m}} (|z_K|^2) \sum_J \tilde{\omega}_J (\bar{z}_J \partial_{\bar{z}_J} \mathbf{Z}^{\mathbf{m}} - z_J \partial_{z_J} \mathbf{Z}^{\mathbf{m}}) &+ \sum_J \sum_{\mathbf{m}} \mathbf{Z}^{\mathbf{m}} a'_{J\mathbf{m}} (|z_J|^2) |z_J|^2. \end{aligned}$$

The 2nd term in the r.h.s. is real valued. The 1st term in the r.h.s. is equal to

$$\sum_{\mathbf{m}, K} a_{K\mathbf{m}} (|z_K|^2) \sum_J \tilde{\omega}_J (\bar{z}_J \partial_{\bar{z}_J} \mathbf{Z}^{\mathbf{m}} - z_J \partial_{z_J} \mathbf{Z}^{\mathbf{m}}) = \sum_{\mathbf{m}, K} a_{K\mathbf{m}} (|z_K|^2) \mathbf{Z}^{\mathbf{m}} \sum_{J, L} \tilde{\omega}_J (m_{LJ} - m_{JL}) = 0.$$

where we have used (2.11) for the last equality.

For

$$G_L(\zeta) := \sum_{(\mu, \nu) \in M_L} \zeta^\mu \bar{\zeta}^\nu G_{\mu\nu} \quad (6.19)$$

we conclude

$$\begin{aligned} \sum_{J=1}^{2n} \frac{d}{dt} A_J(|\zeta_J|^2) &= \frac{1}{4} \sum_{L \in \Lambda} \tilde{\omega} \cdot (\nu - \mu) \operatorname{Im} (G_L |R_{-\sigma_3 B}^+(L) \sigma_3 G_L) + \sum_{J=1}^{2n} \tilde{\omega}_J \operatorname{Im} (\mathcal{G}_J \bar{\zeta}_J) \\ &= \frac{1}{4} \sum_{L \in \Lambda} L \operatorname{Im} (G_L |R_{-\sigma_3 B}^+(L) \sigma_3 G_L) + \sum_{J=1}^{2n} \tilde{\omega}_J \operatorname{Im} (\mathcal{G}_J \bar{\zeta}_J). \end{aligned}$$

We can now substitute $R_{-\sigma_3 B}^+(L) = P.V. \frac{1}{-\sigma_3 B - L} + i\pi \delta(-\sigma_3 B - L)$, which can be defined in terms of distorted Fourier transform associated to $-\Delta + V$, see [29]. The contribution of the principal value cancels out because $P.V. \frac{1}{-\sigma_3 B - L} \sigma_3$ is symmetric. Therefore, we have

$$\begin{aligned} \sum_{J=1}^{2n} \frac{d}{dt} A_J(|\zeta_J|^2) &= -\frac{\pi}{4} \sum_{L \in \Lambda} L \langle G_L | \delta(-\sigma_3 B - L) \sigma_3 G_L \rangle + \sum_{J=1}^{2n} \tilde{\omega}_J \operatorname{Im} (\mathcal{G}_J \bar{\zeta}_J) \\ &= \frac{\pi}{4} \left(\sum_{\substack{L \in \Lambda \\ L > m}} L \langle \pi_2 G_L | \delta(B - L) \pi_2 G_L \rangle + \sum_{\substack{L \in \Lambda \\ L < -m}} |L| \langle \pi_1 G_L | \delta(B - |L|) \pi_1 G_L \rangle \right) + \sum_{J=1}^{2n} \tilde{\omega}_J \operatorname{Im} (\mathcal{G}_J \bar{\zeta}_J), \end{aligned} \quad (6.20)$$

where π_j is the projection to the j -th component and we have used

$$\delta(-\sigma_3 B - L) \sigma_3 = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & -\delta(B - L) \end{pmatrix} & \text{if } L > m, \\ \begin{pmatrix} \delta(-B - L) & 0 \\ 0 & 0 \end{pmatrix} & \text{if } L < -m, \end{cases}$$

and $\delta(-B - L) = \delta(B + L)$. We claim

$$\langle \delta(B - |L|) u | u \rangle \geq 0. \quad (6.21)$$

Let $\kappa \in \mathbb{R}$, $\tau \in \mathbb{R}_+$ and $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ smooth strictly monotonic and with $f(\tau^2) = \kappa$. Then for $H = (-\Delta + V)P_c$, and for \hat{u} and \hat{v} distorted Fourier transforms associated to H we have, ch.2 [13],

$$\begin{aligned} \langle \delta(f(H) - \kappa) u | v \rangle &= \langle \delta(f(\xi^2) - \kappa) \hat{u} | \hat{v} \rangle = \int_{\mathbb{R}} dt \delta(t - \tau) \int_{f(\xi^2)=t} \hat{u}(\xi) \bar{\hat{v}}(\xi) \frac{dS_t}{2f'(t^2)t} \\ &= \frac{1}{2f'(\tau^2)\tau} \int_{|\xi|=\tau} \hat{u}(\xi) \bar{\hat{v}}(\xi) dS \text{ with } dS = dS_\tau, \end{aligned}$$

with dS_t the standard measure on the sphere $|\xi| = t$ in \mathbb{R}^3 . For $L > m$ and $f(t) = \sqrt{t + m^2}$ we have

$$\langle \delta(B - |L|) u | u \rangle = \frac{|L|}{\sqrt{L^2 - m^2}} \int_{|\xi|=\sqrt{L^2 - m^2}} |\hat{u}|^2 dS \geq 0.$$

We are finally able to state in a precise way hypothesis (H4).

(H4) We assume that there is a fixed constant $\mathfrak{c} > 0$ s.t. for all $\zeta \in \mathbb{C}^{2n}$ with $|\zeta| \leq 1$,

$$\sum_{\substack{L \in \Lambda \\ L > m}} L \langle \pi_2 G_L | \delta(B - L) \pi_2 G_L \rangle + \sum_{\substack{L \in \Lambda \\ L < -m}} |L| \langle \pi_1 G_L | \delta(B - |L|) \pi_1 G_L \rangle \geq \frac{4\mathfrak{c}}{\pi} \sum_{(\mu, \nu) \in M_{\min}} |\zeta^{\mu+\nu}|^2. \quad (6.22)$$

By an application of Lemma 6.7, (6.20) and (H4) we arrive as in [9] at

$$\sum_{J=1}^{2n} \frac{d}{dt} A_J(|\zeta_J|^2) \geq \mathfrak{c} \sum_{(\mu, \nu) \in M_{min}} |\zeta^{\mu+\nu}|^2 + \sum_{J=1}^{2n} \tilde{\omega}_J \operatorname{Im}(\mathcal{G}_J \bar{\zeta}_J).$$

Using $A_J(|\zeta_J(t)|^2) = 4\omega_J^2 |\zeta_J(t)|^2 + O(|\zeta_J(t)|^4)$, (6.15), going back to the z and for c_1 the constant in (6.18), when we integrate the above equation, we get

$$4 \sum_{J=1}^2 \omega_J^2 (|z_J(0)|^2 - |z_J(t)|^2) + \mathfrak{c} \sum_{(\mu, \nu) \in M_{min}} \|z^{\mu+\nu}\|_{L^2(0,t)}^2 \leq 3(1 + C_0)c_1\epsilon^2.$$

On the other hand, by the conservation of the energy (5.6) we have $|z(t)| \lesssim \epsilon$ for a some fixed constant. So we can conclude, perhaps for a larger c_1 , that

$$\sum_{(\mu, \nu) \in M} \|z^{\mu+\nu}\|_{L^2(0,t)}^2 \leq 8\mathfrak{c}^{-1}c_1(1 + C_0)\epsilon^2.$$

This tells us that $\|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim C_0^2\epsilon^2$ implies $\|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim \epsilon^2 + C_0\epsilon^2$ for all $(\mu, \nu) \in M_{min}$. This means that we can take $C_0 \sim 1$ and completes the proof of Prop. 5.5.

A Proof of (iii) of Lemma 2.2

Lemma A.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ satisfy $f(e^{i\theta}z) = e^{i\theta}f(z)$. Then, we have*

$$\begin{aligned} e^{-i\theta} \partial_{z_R} f(e^{i\theta}z) &= \cos \theta \partial_{z_R} f(z) - \sin \theta \partial_{z_I} f(z), \\ e^{-i\theta} \partial_{z_I} f(e^{i\theta}z) &= \sin \theta \partial_{z_R} f(z) + \cos \theta \partial_{z_I} f(z). \end{aligned}$$

Proof. We set $f_\theta(z) := f(e^{i\theta}z)$. First, we have

$$\begin{aligned} \partial_{z_R} f_\theta(z) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(e^{i\theta}(z + \varepsilon)) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(e^{i\theta}z + \cos \theta \varepsilon + i \sin \theta \varepsilon) \\ &= \cos \theta \partial_{z_R} f(e^{i\theta}z) + \sin \theta \partial_{z_I} f(e^{i\theta}z), \\ \partial_{z_I} f_\theta(z) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(e^{i\theta}(z + i\varepsilon)) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(e^{i\theta}z - \sin \theta \varepsilon + i \cos \theta \varepsilon) \\ &= \sin \theta \partial_{z_R} f(e^{i\theta}z) + \cos \theta \partial_{z_I} f(e^{i\theta}z). \end{aligned}$$

On the other hand,

$$\begin{aligned} \partial_{z_R} f_\theta(z) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(e^{i\theta}(z + \varepsilon)) = e^{i\theta} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(z + \varepsilon) = e^{i\theta} \partial_{z_R} f(z), \\ \partial_{z_I} f_\theta(z) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(e^{i\theta}(z + i\varepsilon)) = e^{i\theta} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(z + i\varepsilon) = e^{i\theta} \partial_{z_I} f(z), \end{aligned}$$

Therefore, we have the conclusion. □

Proof of (iii) of Lemma 2.2. We set $c_{JA} := C'_{JA}(e^{i\theta}z)(e^{i\theta}\Xi)$ and $\partial_{J'R}\Phi[0] = \psi_0$, $\partial_{J'I}\Phi[0] = i\psi_0$. Then, since $C'_{JA}(e^{i\theta}z)(e^{i\theta}\Xi)$ is the unique solution of (2.7) with z, Ξ replaced by $e^{i\theta}z, e^{i\theta}\Xi$, we have

$$\begin{aligned} 0 &= \Omega \left(\partial_{z_{JR}} \Phi[e^{i\theta}z] |B|^{-\frac{\sigma_3}{2}} e^{i\theta} \Xi + \sum_{J'=1}^{2n} c_{J'R} \psi_0 + i c_{J'I} \psi_0 \right) \\ &= \cos \theta \Omega \left(\partial_{z_{JR}} \Phi[z] |B|^{-\frac{\sigma_3}{2}} \Xi + \sum_{J'=1}^{2n} (\cos \theta c_{J'R} + \sin \theta c_{J'I}) \psi_0 + (\cos \theta c_{J'I} - \sin \theta c_{J'R}) i \psi_0 \right) \\ &\quad - \sin \theta \Omega \left(\partial_{z_{JI}} \Phi[z] |B|^{-\frac{\sigma_3}{2}} \Xi + \sum_{J'=1}^{2n} (\cos \theta c_{J'R} + \sin \theta c_{J'I}) \psi_0 + (\cos \theta c_{J'I} - \sin \theta c_{J'R}) i \psi_0 \right) \end{aligned}$$

and

$$\begin{aligned} 0 &= \Omega \left(\partial_{z_{JI}} \Phi[e^{i\theta}z] |B|^{-\frac{\sigma_3}{2}} e^{i\theta} \Xi + \sum_{J'=1}^{2n} c_{J'R} \psi_0 + i c_{J'I} \psi_0 \right) \\ &= \sin \theta \Omega \left(\partial_{z_{JR}} \Phi[z] |B|^{-\frac{\sigma_3}{2}} \Xi + \sum_{J'=1}^{2n} (\cos \theta c_{J'R} + \sin \theta c_{J'I}) \psi_0 + (\cos \theta c_{J'I} - \sin \theta c_{J'R}) i \psi_0 \right) \\ &\quad + \cos \theta \Omega \left(\partial_{z_{JI}} \Phi[z] |B|^{-\frac{\sigma_3}{2}} \Xi + \sum_{J'=1}^{2n} (\cos \theta c_{J'R} + \sin \theta c_{J'I}) \psi_0 + (\cos \theta c_{J'I} - \sin \theta c_{J'R}) i \psi_0 \right), \end{aligned}$$

for all J . Therefore, we have

$$0 = \Omega \left(\partial_{z_{JA}} \Phi[z] |B|^{-\frac{\sigma_3}{2}} \Xi + \sum_{J'=1}^{2n} (\cos \theta c_{J'R} + \sin \theta c_{J'I}) \psi_0 + (\cos \theta c_{J'I} - \sin \theta c_{J'R}) i \psi_0 \right),$$

for all J, A .

Since $C'_{J'A'}(z)\Xi$ is a unique solution of (2.7), we have

$$C'_{J'R}(z)\Xi = \cos \theta c_{J'R} + \sin \theta c_{J'I}, \quad C'_{J'I}(z)\Xi = -\sin \theta c_{J'R} + \cos \theta c_{J'I}.$$

Therefore, we obtain,

$$\begin{aligned} C'_{J'R}(e^{i\theta}z)(e^{i\theta}\Xi) &= \cos \theta C'_{J'R}(z)\Xi - \sin \theta C'_{J'I}(z)\Xi, \\ C'_{J'I}(e^{i\theta}z)(e^{i\theta}\Xi) &= \sin \theta C'_{J'R}(z)\Xi + \cos \theta C'_{J'I}(z)\Xi, \end{aligned}$$

Finally, notice that it suffices to show $R[e^{i\theta}z](e^{i\theta}\Xi) = e^{i\theta}R[z]\Xi$.

$$\begin{aligned} R[e^{i\theta}z](e^{i\theta}\Xi) &= B^{-\frac{\sigma}{2}} e^{i\theta} \Xi + \sum_{J'=1}^{2n} (C'_{J'R}(e^{i\theta}z)e^{i\theta}\Xi) \psi_0 + (C'_{J'I}(e^{i\theta}z)e^{i\theta}\Xi) i \psi_0 = \\ &= e^{i\theta} B^{-\frac{\sigma}{2}} \Xi + \sum_{J'=1}^{2n} (C'_{J'R}(\cos \theta C'_{J'R}(z)\Xi - \sin \theta C'_{J'I}(z)\Xi) \psi_0 + (\sin \theta C'_{J'R}(z)\Xi + \cos \theta C'_{J'I}(z)\Xi) e^{i\theta} \Xi) i \psi_0 \\ &= e^{i\theta} \left(B^{-\frac{\sigma}{2}} \Xi + \sum_{J'=1}^{2n} (C'_{J'R}(z)\Xi) \psi_0 + (C'_{J'I}(z)\Xi) i \psi_0 \right) = e^{i\theta} R[z]\Xi. \end{aligned}$$

Therefore, we have the conclusion. \square

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References

- [1] X.An, A.Soffer, *Remark on Asymptotic Completeness for Nonlinear Klein-Gordon Equations with Metastable States*, arXiv:1510.06485.
- [2] D.Bambusi, S.Cuccagna, *On dispersion of small energy solutions of the nonlinear Klein Gordon equation with a potential*, Amer. Math. Jour. **133** (2011), 1421–1468.
- [3] N.Boussaid, S.Cuccagna, *On stability of standing waves of nonlinear Dirac equations*, Comm. in Partial Diff. Eq. **37** (2012), 1001–1056.
- [4] V.Buslaev, G.Perelman, *Scattering for the nonlinear Schrödinger equation: states close to a soliton*, St. Petersburg Math.J., **4** (1993), pp. 1111–1142.
- [5] V.Buslaev, G.Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations, editor N.N. Uraltseva, Transl. Ser. 2, 164, Amer. Math. Soc., pp. 75–98, Amer. Math. Soc., Providence (1995).
- [6] S.Cuccagna, *On the Darboux and Birkhoff steps in the asymptotic stability of solitons*, Rend. Istit. Mat. Univ. Trieste **44** (2012), 197–257.
- [7] S.Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states*, Comm. Math. Physics **305** (2011), 279–331.
- [8] S.Cuccagna, M.Maeda, *On weak interaction between a ground state and a non-trapping potential*, J. Differential Equations **256** (2014), 1395–1466.
- [9] S.Cuccagna, M.Maeda, *On small energy stabilization in the NLS with a trapping potential*, Anal. PDE **8** (2015), 1289–1349.
- [10] S.Cuccagna, M.Maeda, *On orbital instability of spectrally stable vortices of the NLS in the plane*, arXiv:1508.03146.
- [11] S.Cuccagna, M.Tarulli, *On stabilization of small solutions in the nonlinear Dirac equation with a trapping potential*, arXiv:1309.4878.
- [12] P. D’Ancona, L. Fanelli, *Strichartz and smoothing estimates for dispersive equations with magnetic potentials*, Comm. P.D.E., **33**(6) (2008), pp. 1082–1112.
- [13] F.G. Friedlander, *The wave equation on a curved space-time*, Cambridge Un. Press (1975).
- [14] Zhou Gang, I.M.Sigal, *Relaxation of Solitons in Nonlinear Schrödinger Equations with Potential*, Advances in Math., **216** (2007), pp. 443–490.

- [15] Zhou Gang, M.I.Weinstein, *Dynamics of Nonlinear Schrödinger/Gross-Pitaeski Equations; Mass transfer in Systems with Solitons and Degenerate Neutral Modes*, Anal. PDE **1** (2008), 267–322.
- [16] Zhou Gang, M.I.Weinstein, *Equipartition of Energy in Nonlinear Schrödinger/Gross-Pitaeski Equations*, Appl. Math. Res. Express. AMRX **2** (2011), 123–181.
- [17] M.Grillakis, J.Shatah, W.Strauss, *Stability of solitary waves in the presence of symmetries, I*, Jour. Funct. An. **74** (1987), pp.160–197.
- [18] S.Gustafson, K.Nakanishi, T.P.Tsai, *Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equations with small solitary waves*, Int. Math. Res. Not. 2004 (2004) no. **66**, 3559–3584.
- [19] S.Gustafson, T.V.Phan, *Stable directions for degenerate excited states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **43** (2011), 1716–1758.
- [20] M.Maeda, *Existence and asymptotic stability of quasi-periodic solution of discrete NLS with potential in \mathbb{Z}* , arXiv:1412.3213.
- [21] K.Nakanishi, *Global dynamics below excited solitons for the nonlinear Schrödinger equation with a potential*, arXiv:1504.06532.
- [22] K.Nakanishi, T.V.Phan, T.P.Tsai, *Small solutions of nonlinear Schrödinger equations near first excited states*, Jour. Funct. Analysis **263** (2012), 703–781.
- [23] D.Pelinovsky, A.Stefanov. *Asymptotic stability of small gap solitons in the nonlinear dirac equations*, J. Math. Phys. **53** (2012), 073705, 27 pp.
- [24] C.A.Pillet, C.E.Wayne, *Invariant manifolds for a class of dispersive, Hamiltonian partial differential equations* J. Diff. Eq. **141** (1997), pp. 310–326.
- [25] A.Soffer, M.I.Weinstein, *Multichannel nonlinear scattering for nonintegrable equations*, Comm. Math. Phys., **133** (1990), pp. 116–146
- [26] A.Soffer, M.I.Weinstein, *Multichannel nonlinear scattering II. The case of anisotropic potentials and data*, J. Diff. Eq., **98** (1992), pp. 376–390.
- [27] A.Soffer, M.I.Weinstein, *Selection of the ground state for nonlinear Schrödinger equations*, Rev. Math. Phys. **16** (2004), 977–1071.
- [28] A.Soffer, M.I.Weinstein, *Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations*, Invent. Math. **136** (1999), 9–74.
- [29] M.E. Taylor, *Partial Differential Equations II*, App. Math. Sci. **116**, Springer, New York (1997).
- [30] T.P.Tsai, H.T.Yau, *Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and radiation dominated solutions*, Comm. Pure Appl. Math. **55** (2002), 153–216.
- [31] T.P.Tsai, H.T.Yau, *Relaxation of excited states in nonlinear Schrödinger equations*, Int. Math. Res. Not. **31** (2002), 1629–1673.

- [32] T.P.Tsai, H.T.Yau, *Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data*, Adv. Theor. Math. Phys. **6** (2002), 107–139.
- [33] T.P.Tsai, H.T.Yau, *Stable directions for excited states of nonlinear Schrödinger equations*, Comm. P.D.E. **27** (2002), 2363–2402.
- [34] K.Yajima, *The $W^{k,p}$ -continuity of wave operators for Schrödinger operators*, J. Math. Soc. Japan, 47 (1995), pp. 551–581.

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